Orthonormal Basis for the Column Space and the Null Space
using the QR Decomposition

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1 Introduction

The QR Decomposition can be used to get orthonormal bases for the column space $C(A)$ and the left null space $N(A^\top)$.

2 QR Decomposition

The QR Decomposition of a matrix $A$ is $A = QR$, where $Q$ is a $m \times m$ orthogonal matrix, $R$ is a $m \times n$ upper triangular matrix. The matrix $A$ can also be split into the blocks as follows:

$$A = QR = \begin{bmatrix} Q_1 & Q_2 \\ \end{bmatrix} \begin{bmatrix} R_1 \\ 0 \\ \end{bmatrix}$$

where $Q_1 = Q^{\top}_1 Q_1 = I_r$ and $Q_2 = Q^{\top}_2 Q_2 = I_{m-r}$.

3 Basis Sets

3.1 Column Space

Theorem 1. Suppose $A$ is any $m \times n$ matrix, and $A = QR$ is the full rank QR Decomposition. It follows that an orthonormal set of basis vectors for $C(A)$, the column space, are the columns of $Q$. 
Proof. Suppose $b \in C(A)$. It follows that

$$b = Az = Q_1 R_1 z = Q_1 z^*$$

Therefore, $b \in C(Q_1)$.

Since the columns of $Q_1$ are orthonormal, then they are linearly independent to each other. Since $\dim C(A) = r$ and there are $r$ linearly independent columns in $Q_1$, then it follows that the columns of $Q_1$ form a basis for $C(A)$. It follows that $C(A) = C(Q_1)$. \hfill \square

3.2 Left Null Space

**Theorem 2.** Suppose $A$ is any $m \times n$ matrix, and $A = QR$ is the QR Decomposition. Partition $Q$ into $[Q_1 \quad Q_2]$, where $Q_1$ is the first $r$ columns of $Q$, $Q_2$ is the last $m - r$ columns of $Q$, and $r$ is the rank of $A$. It follows that an orthonormal set of basis vectors for $N(A^\top)$, the left null space, are the columns of $Q_2$.

Proof. Solve equation (1) for $R$ results with

$$R = \begin{bmatrix} R_1 \\ 0 \end{bmatrix} = \begin{bmatrix} Q_1^\top \\ Q_2^\top \end{bmatrix} A = \begin{bmatrix} Q_1^\top A \\ Q_2^\top A \end{bmatrix}$$

(3)

It follows that

$$Q_2^\top A = 0 \implies A^\top Q_2 = 0$$

It follows that $Q_2 \in N(A^\top)$. Since the columns of $Q_2$ are orthonormal, then they are linearly independent of each other. Since $\dim N(A^\top) = m - r$ and there are $m - r$ columns in $Q_2$, then it follows that the columns of $Q_2$ form a basis for $N(A^\top)$. \hfill \square

4 Summary

Define the QR Decomposition as

$$A = QR = \begin{bmatrix} Q_1 & Q_2 \end{bmatrix} \begin{bmatrix} R_1 \\ 0 \end{bmatrix}$$

It follows that

$Q_1$ forms a basis for $C(A)$

$Q_2$ forms a basis for $N(A^\top)$