

8.4 The t, F, and Beta Dist.

Certain functions of normal samples are very important.

Student's t dist.

Motivation

Note: s^2 can be used to make inferences about σ^2
 \bar{X} can be used to make inferences about μ .

$$\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1)$$

puts

Thm: If $Z \sim N(0, 1)$
 $C \sim \chi^2(\nu)$
 $Z \perp\!\!\!\perp C$, then

$$T = \frac{Z}{\sqrt{\frac{C}{\nu}}}$$

is referred to as Student's t dist with degrees of freedom ν .

$$T \sim t(\nu)$$

Proof: The joint density of Z, C is $Z \sim N(0,1)$ $C \sim \chi^2(\nu)$

$$f_{Z,C}(z,c) = \frac{1}{2^{\nu/2} \Gamma(\frac{\nu}{2}) \sqrt{2\pi}} c^{\frac{\nu}{2}-1} e^{-\frac{1}{2}(z^2+c)}$$

Make the transform $T = \frac{z}{\sqrt{c}}$, $W = C$

$z = T \sqrt{\frac{W}{\nu}}$, $C = W$

$$|J| = \begin{vmatrix} \frac{\partial C}{\partial W} & \frac{\partial C}{\partial T} \\ \frac{\partial z}{\partial W} & \frac{\partial z}{\partial T} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \frac{1}{\sqrt{\nu}} & \frac{T}{\sqrt{W}} \end{vmatrix} = \sqrt{\frac{W}{\nu}}$$

$$f_{T,W}(t,w) = f_{Z,C}(z,c) |J|$$

$$= f_{Z,C}\left(t\sqrt{\frac{w}{\nu}}, w\right) \sqrt{\frac{w}{\nu}}$$

$$f_{T,W}(t,w) = c \cdot w^{\frac{\nu}{2}-1} e^{-\frac{1}{2}\left(\frac{t^2}{\nu}w+w\right)} \sqrt{\frac{w}{\nu}}$$

$$f_T(t) = \int_0^{\infty} f_{T,W}(t,w) dw$$

$$= \frac{c}{\sqrt{\nu}} \int_0^{\infty} w^{\left(\frac{\nu}{2}-1\right)} e^{-\frac{1}{2}w\left(\frac{t^2}{\nu}+1\right)} dw$$

$$u = \frac{1}{2}\left(\frac{t^2}{\nu}+1\right)w$$

$$du = \frac{1}{2}\left(\frac{t^2}{\nu}+1\right) dw$$

Snedecor's F dist

Thm: If $V_1 \sim \chi^2(v_1)$ and $V_2 \sim \chi^2(v_2)$,

and $V_1 \perp V_2$, then

$$F = \frac{(V_1/v_1)}{(V_2/v_2)} = \frac{V_1}{v_1} \cdot \frac{v_2}{V_2} = \frac{v_2}{v_1} V_1 V_2^{-1}$$

is distributed as Snedecor's F dist
with v_1 and v_2 degrees of freedom

v_1 = numerator degrees of freedom,

v_2 = denominator degrees of freedom. $-\left(\frac{v_1+v_2}{2}\right)$

$$f(x) = \frac{\Gamma\left(\frac{v_1+v_2}{2}\right)}{\Gamma\left(\frac{v_1}{2}\right)\Gamma\left(\frac{v_2}{2}\right)} \left(\frac{v_1}{v_2}\right)^{\frac{v_1}{2}} x^{\frac{v_1}{2}-1} \left(1 + \frac{v_1}{v_2}x\right)^{-\left(\frac{v_1+v_2}{2}\right)}$$

Thm: If $X \sim F(v_1, v_2)$,

then

$$E[X^r] = \frac{\left(\frac{v_2}{v_1}\right)^r \Gamma\left(\frac{v_1}{2} + r\right) \Gamma\left(\frac{v_2}{2} - r\right)}{\Gamma\left(\frac{v_1}{2}\right) \Gamma\left(\frac{v_2}{2}\right)}$$

$$v_2 > 2r$$

$$E[X] = \frac{v_2}{v_2 - 2}, \quad v_2 > 2.$$

$$\text{Var}(X) = \frac{2v_2^2(v_1 + v_2 - 2)}{v_1(v_2 - 2)^2(v_2 - 4)}, \quad v_2 > 4$$

Ex. Let X_1, X_2, \dots, X_{n_1}
 Y_1, Y_2, \dots, Y_{n_2}
from a pop with

indep R.S.

$$X_i \sim N(\mu_1, \sigma_1^2)$$

$$Y_j \sim N(\mu_2, \sigma_2^2)$$

$$v_1 = n_1 - 1, \quad v_2 = n_2 - 1$$

Then

$$\frac{v_1 S_1^2}{\sigma_1^2} \sim \chi^2(v_1) \quad \frac{v_2 S_2^2}{\sigma_2^2} \sim \chi^2(v_2)$$

$$\frac{\left(\frac{v_1 S_1^2}{\sigma_1^2}\right) / v_1}{\left(\frac{v_2 S_2^2}{\sigma_2^2}\right) / v_2} = \frac{\sigma_2^2 S_1^2}{\sigma_1^2 S_2^2} \sim F(v_1, v_2)$$

Beta Distribution

If $X \sim F(v_1, v_2)$, then

$$Y = \frac{\frac{v_1}{v_2} X}{1 + \frac{v_1}{v_2} X} \sim \text{BETA}\left(\frac{v_1}{2}, \frac{v_2}{2}\right)$$

If $W \sim \text{BETA}(\alpha, \beta)$

$$f_W(w) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} w^{\alpha-1} (1-w)^{\beta-1}$$

$$\times I_{(0,1)}(w)$$