

### 8.3 Sampling Dist.

A statistic is a random var.

It has a dist which depends on the R.S.  
this dist is called the sampling dist.

### Linear Combinations of Normal R.V.

Thm: If  $X_i \sim N(\mu_i, \sigma_i^2)$  denote independent R.V. It follows that

$$Y = \sum_{i=1}^n a_i X_i \sim N\left(\sum_{i=1}^n a_i \mu_i, \sum_{i=1}^n a_i^2 \sigma_i^2\right)$$

Proof: Use MGF's.

Corollary:

If  $X_1, \dots, X_n$  is RS  $N(\mu, \sigma^2)$

then

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

Note: compare to CLT

$$\frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} \xrightarrow{d} N(0, 1)$$

$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

Chi-Squared dist

"chi" is pronounced like "Kite"

$\chi^2$

If  $Y \sim \text{GAM}\left(2, \frac{\nu}{2}\right)$ , the  $Y$  is said to have a Chi-Squared dist.  
 $\nu = \text{degrees of freedom}$

nu  $\nu$ , not  $\nu$ , but  $\nu$

$$Y \sim \chi^2(\nu)$$

Thm. If  $Y \sim \chi^2(\nu)$

$$M_Y(t) = (1-2t)^{-\nu/2}$$

$$E[Y^r] = \frac{2^r \Gamma(\frac{\nu}{2} + r)}{\Gamma(\frac{\nu}{2})}, r > -\frac{\nu}{2}$$

$$E(Y) = \nu$$

$$V(Y) = 2\nu$$

Thm. If  $X \sim \text{GAM}(\theta, k)$ , then

$$Y = \frac{2X}{\theta} \sim \chi^2(2k)$$

We can use a  $\chi^2$ -table to compute the CDF for any Gamma dist.

$$F_X(x) = F_Y\left(\frac{2x}{\theta}, 2k\right)$$

Ex:

$X =$  time in years until failure of a comp.

$$X \sim \text{GAM}(\theta, k)$$

Find the 10<sup>th</sup> Percentile.

$$P[X \leq \underbrace{X_{0.10}}_{\text{10th percentile}}] = F_Y\left(\frac{2X_{0.10}}{\theta}\right) = .10$$

$$\frac{2X_{0.10}}{\theta} = \chi_{0.10}^2(2k)$$

$$\frac{2}{3}X_{0.10} = \chi_{0.10}^2(4) = 1.06$$

$$X_{0.10} = \frac{3}{2}(1.06) = 1.59$$

Thm:

If  $Y_i \sim \chi^2(v_i)$  and  $Y_i$  are ind

then

$$V = \sum_{i=1}^n Y_i \sim \chi^2\left(\sum_{i=1}^n v_i\right)$$

Proof: Use MGF.

Thm:

If  $Z \sim N(0, 1)$ ,

then  $Z^2 \sim \chi^2(1)$

Proof:

Transform to  $Y = Z^2$

Corollary IF  $X_1, \dots, X_n$  is a RS from  $N(\mu, \sigma^2)$

$$\sum_{i=1}^n \frac{(X_i - \mu)^2}{\sigma^2} \sim \chi^2(n)$$

$$\frac{n(\bar{X} - \mu)^2}{\sigma^2} \sim \chi^2(1)$$

Proof:

$$\bar{X} \sim N(\mu, \frac{\sigma^2}{n}) \Rightarrow \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0, 1) \Rightarrow \left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}\right)^2 \sim \chi^2(1)$$

linear comb of normal R.V.s.

$$Y = \sum a_i X_i \sim N(\sum a_i \mu_i, \sum a_i^2 \sigma_i^2)$$
$$\bar{X} = \frac{1}{n} \sum X_i \Rightarrow a_i = \frac{1}{n}$$

$$\parallel$$
$$\frac{n(\bar{X} - \mu)^2}{\sigma^2}$$

What is the dist of

$$S^2 = \sum_{i=1}^n \frac{(X_i - \bar{X})^2}{n-1}$$

Difficult  $X_i - \bar{X}$  is not indep of  $X_j - \bar{X}$ .

$$\sum (X_i - \bar{X}) = 0.$$

Thm: If  $X_1, \dots, X_n$  denote a RS from  $N(\mu, \sigma^2)$

1.  $\bar{X}$  and  $X_i - \bar{X}$  are independent
2.  $\bar{X}$  and  $S^2$  are independent
3.  $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$