

7.7

## Additional Limit Thms

### DEF. Convergence in Probability

The sequence of random vars  $Y_n$  is said to converge in probability to  $Y$ , written

$$Y_n \xrightarrow{P} Y \quad \text{if}$$

$$\lim_{n \rightarrow \infty} P[|Y_n - Y| < \varepsilon] = 1$$

Note: this is a bit more general than stochastic convergence to a constant. Most of the time we talk about when  $Y$  is constant.

Note that convergence in prob. is stronger than convergence in distribution.

Thm: If  $Y_n \xrightarrow{P} Y$ , then  $Y_n \xrightarrow{d} Y$ .

Compare to last result in section 7.6.

If  $Z_n = \frac{\sqrt{n}(Y_n - m)}{c} \xrightarrow{d} N(0,1)$ , then  $Y_n \xrightarrow{P} m$ .

For the special case of  $Y=c$ ,  
the limiting dist is the  
degenerate dist  $P[Y=c]=1$

Thm: If  $Y_n \xrightarrow{P} c$ , then for any function  
 $g(y)$  that is continuous at  $c$ .

$$g(Y_n) \xrightarrow{P} g(c).$$

Proof: Because  $g(y)$  is cont. at  $c$ , then

$\forall \epsilon > 0, \exists \delta > 0$ , such that

$$|y-c| < \delta \Rightarrow |g(y)-g(c)| < \epsilon$$

$\Rightarrow$

$$P[|g(Y_n)-g(c)| < \epsilon] \geq P[|Y_n-c| < \delta]$$

Because  $Y_n \xrightarrow{P} c$ , then

$$\lim_{n \rightarrow \infty} P[|g(Y_n)-g(c)| < \epsilon] \geq \lim_{n \rightarrow \infty} P[|Y_n-c| < \delta] = 1$$

Therefore,

$$\lim_{n \rightarrow \infty} P[|g(Y_n)-g(c)| < \epsilon] = 1$$

$$g(Y_n) \xrightarrow{P} g(c)$$

Thm: If  $X_n \neq Y_n$  two seq. R.V.

$$X_n \xrightarrow{P} c$$

$$Y_n \xrightarrow{P} d.$$

$$(1.) aX_n + bY_n \xrightarrow{P} ac + bd$$

$$(2.) X_n Y_n \xrightarrow{P} cd$$

$$(3.) \frac{X_n}{c} \xrightarrow{P} 1, c \neq 0.$$

$$(4.) \frac{1}{X_n} \xrightarrow{P} \frac{1}{c}, c \neq 0, P[X_n = 0] = 0$$

$$(5.) \sqrt{X_n} \xrightarrow{P} \sqrt{c}, P[X_n \geq 0] = 1, c > 0$$

Ex:  $Y \sim \text{BIN}(n, p)$

$$\hat{p} = \frac{Y}{n} \xrightarrow{P} p \quad (\text{Bernolli Law of Large \#s})$$

$$\hat{p}(1-\hat{p}) \xrightarrow{P} p(1-p)$$

Thm: Slutsky's Thm

$X_n, Y_n$  R.V.

$$X_n \xrightarrow{P} c, Y_n \xrightarrow{d} Y,$$

$$1. X_n + Y_n \xrightarrow{d} c + Y$$

$$2. X_n Y_n \xrightarrow{d} cY$$

$$3. \frac{Y_n}{X_n} \xrightarrow{d} \frac{Y}{c}, c \neq 0.$$

Ex:  $X_i \sim \text{BIN}(1, p)$  RS. of size  $n$ .

$$\frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}} \xrightarrow{d} Z \sim N(0, 1) \quad (\text{by CLT})$$

We also know  $\hat{p}(1-\hat{p}) \xrightarrow{P} p(1-p)$ .

$$\frac{\hat{p} - p}{\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}} \xrightarrow{d} Z \sim N(0, 1)$$

Thm: If  $Y_n \xrightarrow{d} Y$ , then for any cont. function  $g(y)$ ,

$$g(Y_n) \xrightarrow{d} g(Y).$$

$g(y)$  is assumed to not depend on  $n$ .

Thm: If  $\frac{\sqrt{n}(Y_n - m)}{c} \xrightarrow{d} Z \sim N(0, 1)$ ,

and if  $g'(m) \neq 0$ , then

$$\frac{\sqrt{n}(g(Y_n) - g(m))}{|c g'(m)|} \xrightarrow{d} Z \sim N(0, 1)$$

Hence,

$$g(\bar{Y}_n) \sim N \left\{ g(\mu), \frac{C^2[g'(\mu)]^2}{n} \right\}$$

EX: By the CLT,

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} Z \sim N(0, 1)$$

$$\bar{X}_n \sim N \left\{ \mu, \frac{\sigma^2}{n} \right\}$$

$$g(x) = x^2 \Rightarrow g'(x) = 2x$$

It follows that

$$\bar{X}_n^2 \sim N \left\{ \mu^2, \frac{\sigma^2(2\mu)^2}{n} = \frac{4\sigma^2\mu^2}{n} \right\}$$