7.5 Asymptotic Normal Dist.

DEF: If $Y_1, Y_2, \dots$ is a sequence of random variables and m, c are constants such that

$$Z_n = \frac{Y_n - m}{c/\sqrt{n}} \xrightarrow{d} Z \sim N(0,1)$$

as $n \to \infty$, then Y_n is said to have an asymptotic normal dist with asymptotic mean m, and asymptotic variance c^2/n

Ex: Ex 4.6.3.

$n = 40$ lifetimes of electrical parts

$X_i \sim EXP(100)$

By CLT,

$\bar{X}_n$ has an asymptotic normal dist with mean $m = 100$, and variance $\frac{c^2}{n} = \frac{100^2}{40}$

$= 250$

Thm 7.5.1.

$X_1, \dots, X_n$ RS from cont. dist.

x_p is the p^{th} percentile

If $\frac{k}{n} \to p$, then the sequence of kth order stats $X_{k:n}$ is asymptotically normal with mean x_p and variance c^2/n, where

$$c^2 = \frac{p(1-p)}{[f(x_p)]^2}$$

$$\frac{X_{k:n} - x_p}{c/\sqrt{n}} \xrightarrow{d} N(0,1)$$

7.6 Properties of Stochastic Convergence

Sequence of r.v. that converge stochastically to a constant.

Thm: The sequence $Y_1, Y_2, \dots$ converges stochastically to c iff $\forall \varepsilon > 0$, $\lim_{n\to\infty} P[|Y_n - c| < \varepsilon] = 1$.

If a sequence of R.V. satisfies * then the sequence is said to converge in probability to c.
In statistical notation,

$$Y_n \xrightarrow{P} c.$$

Ex: Bernoulli Law of Large numbers.

Use Chebychev's Inequality.

$$P[|X-\mu| < \varepsilon] \geq 1 - \frac{\sigma^2}{\varepsilon^2}$$

Let $X_1, \dots, X_n$ RS from BERNOULLI(p)

$$\hat{p}_n = \frac{1}{n} \sum X_i$$

$$E(\hat{p}_n) = \frac{1}{n} \sum E(X_i) = \frac{np}{n} = p$$

$$V(\hat{p}_n) = \frac{1}{n^2} \sum V(X_i) = \frac{npq}{n^2} = \frac{pq}{n}$$

It follows that

$$P[|\hat{p}_n - p| < \varepsilon] \geq 1 - \frac{pq/n}{\varepsilon^2} = 1 - \frac{pq}{n\varepsilon^2}$$

$$\lim_{n\to\infty} P[|\hat{p}_n - p| < \varepsilon] \geq \lim_{n\to\infty} 1 - \frac{pq}{n\varepsilon^2} = 1.$$

Thus $\lim_{n\to\infty} P[|\hat{p}_n - p| < \varepsilon] = 1.$

Therefore, $\hat{p}_n \xrightarrow{P} p$

Thm: Law of Large Numbers.

If $X_1, X_2, \cdots, X_n$ is a RS from a dist with finite mean μ, and variance, the the sequence of sample means converge in prob to μ.

$$\bar{X}_n \xrightarrow{P} \mu$$

Proof:

$$E(\bar{X}) = E\left[\frac{1}{n}\sum X_i\right] = \frac{1}{n}\sum E(X_i) = \frac{n\mu}{n} = \mu$$

$$V(\bar{X}) = V\left(\frac{1}{n}\sum X_i\right) = \frac{1}{n^2}\sum V(X_i) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

$$P\left[|\bar{X}-\mu| < \varepsilon\right] \geq 1 - \frac{\sigma^2}{n\varepsilon^2}$$

$$\lim_{n\to\infty} P\left[|\bar{X}-\mu| < \varepsilon\right] \geq \lim_{n\to\infty} 1 - \frac{\sigma^2}{n\varepsilon^2} = 1$$

Thus,

$$\bar{X} \xrightarrow{P} \mu.$$

If $Z_n = \dfrac{\sqrt{n}(Y_n - m)}{c} \xrightarrow{d} Z \sim N(0,1)$,

then $Y_n \xrightarrow{P} m$.

Ex: Bernoulli Law of Large numbers.

Use Chebychev's Inequality.

$$P\left[|X-\mu|<\varepsilon\right] \geq 1-\frac{\sigma^2}{\varepsilon^2}$$

Let $X_1, \dots, X_n$ RS from BERNOULLI(p)

$$\hat{p}_n = \frac{1}{n}\sum X_i$$

$$E(\hat{p}_n) = \frac{1}{n}\sum E(X_i) = \frac{np}{n} = p$$

$$V(\hat{p}_n) = \frac{1}{n^2}\sum V(X_i) = \frac{npq}{n^2} = \frac{pq}{n}$$

It follows that

$$P\left[|\hat{p}_n-p|<\varepsilon\right] \geq 1-\frac{pq/n}{\varepsilon^2} = 1-\frac{pq}{n\varepsilon^2}$$

$$\lim_{n\to\infty} P\left[|\hat{p}_n-p|<\varepsilon\right] \geq \lim_{n\to\infty} 1-\frac{pq}{n\varepsilon^2} = 1.$$

Thus $\lim_{n\to\infty} P\left[|\hat{p}_n-p|<\varepsilon\right] = 1.$

Therefore, $\hat{p}_n \xrightarrow{P} p$