[7.4] Approx. for the Binomial Dist.

Ex: Let $X_1, \dots, X_n$ be a RS from a Bernoulli

$$X_i \sim BIN(1, p)$$

Consider $Y_n = \sum_{i=1}^{n} X_i$

Let $p \to 0$ as $n \to \infty$ in such a way that $\mu = np$ is fixed.

$$M_{Y_n}(t) = (pe^t + q)^n = \left[\frac{\mu}{n} e^t + 1 - \frac{\mu}{n}\right]^n$$

$$= \left(1 + \frac{\mu(e^t - 1)}{n}\right)^n \longrightarrow e^{\mu(e^t-1)}$$

$$Y_n \xrightarrow{d} Y \sim POI(\mu)$$

Ex: Same description as above

$$W_n = \hat{p}_n = \frac{1}{n} Y_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

$$M_{W_n}(t) = E\left[e^{\frac{tY_n}{n}}\right] = M_{Y_n}\left(\frac{t}{n}\right)$$

$$= \left(pe^{\frac{t}{n}} + q\right)^n$$

$$= \left[p\left(1 + \frac{t}{n} + \frac{t^2}{2n^2} + \frac{t^3}{3!n^3} + \cdots\right) + 1 - p\right]^n$$

$$= \Big[1 + \frac{pt}{n} + \underbrace{\frac{pt^2}{2n^2} + \frac{pt^3}{3!n^3} + }_{\frac{d(n)}{n}}\Big]^n$$

$$\to e^{pt}$$

Note : $M_{W_n}(t) = e^{pt}$

This is the MGF for a degenerate dist at $y=p$.

$$f(y) = \begin{cases} 1, & y=p \\ 0, & \text{otherwise.} \end{cases}$$

$Y \sim \text{DEGENERATE}(p)$.

$$M_Y(t) = E\left[e^{tY}\right] = \sum_{\text{all } y} e^{ty} f(y) = e^{pt} f(p) = e^{pt}$$

So y converges stochastically to p as $n \to \infty$.

EX: Consider the "Standardized" Variable

$$Z_n = \frac{Y_n - np}{\sqrt{npq}} = \frac{Y_n - np}{\sigma_n}$$

$$M_{Z_n}(t) = E\left[e^{tZ_n}\right] = E\left[\exp\left\{t\,\frac{Y_n - np}{\sigma_n}\right\}\right]$$

$$= e^{-\frac{npt}{\sigma_n}} E\left[e^{\frac{t}{\sigma_n}Y_n}\right] = e^{-\frac{npt}{\sigma_n}} M_{Y_n}\left(\frac{t}{\sigma_n}\right)$$

$$= e^{-\frac{npt}{\sigma_n}}\left(pe^{\frac{t}{\sigma_n}} + q\right)^n$$

$$= \left[e^{-\frac{pt}{\sigma_n}}\left(pe^{t/\sigma_n} + q\right)\right]^n$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{-pt/\sigma_n} = 1 - \frac{p}{\sigma_n}t + \frac{1}{2}\frac{p^2}{\sigma_n^2}t^2 - \frac{1}{6}\frac{p^3}{\sigma_n^3}t^3 + \cdots$$

$$e^{t/\sigma_n} = 1 + \frac{t}{\sigma_n} + \frac{1}{2\sigma_n^2}t^2 + \frac{1}{6\sigma_n^3}t^3 + \cdots$$

$$pe^{t/\sigma_n} + q = 1 - p + p + \frac{pt}{\sigma_n} + \frac{pt^2}{2\sigma_n^2} + \frac{pt^3}{6\sigma_n^3} + \cdots$$

$$= 1 + \frac{pt}{\sigma_n} + \frac{pt^2}{2\sigma_n^2} + \frac{pt^3}{6\sigma_n^3} + \cdots$$

$$e^{-pt/\sigma_n}\left(pe^{t/\sigma_n} + q\right)$$

$$= \left(1 - \frac{p}{\sigma_n}t + \frac{1}{2}\frac{p^2}{\sigma_n^2}t^2 - \frac{1}{6}\frac{p^3}{\sigma_n^3}t^3 + \cdots\right) \times$$

$$\left(1 + \frac{pt}{\sigma_n} + \frac{pt^2}{2\sigma_n^2} + \frac{pt^3}{6\sigma_n^3} + \cdots\right)$$

$$\sigma_n^k = \left(\sqrt{npq}\right)^k = n^{k/2}(pq)^{k/2}$$

$$\sigma_n^2 = npq = O(n) \qquad \sigma_n^3 = O(n^{3/2})$$

$$= 1 + \cancel{\frac{pt}{\sigma_n}} - \cancel{\frac{pt}{\sigma_n}} - \frac{p^2t^2}{\sigma_n^2} + \frac{1}{2}\frac{pt^2}{\sigma_n^2} + \frac{p^2t^2}{2\sigma_n^2} + \frac{1}{\sigma_n^3}(\text{stuff})$$

$$= \left[1 + \frac{t^2}{\sigma_n^2}\left(\frac{p}{2} - p^2 + \frac{p^2}{2}\right) + \frac{(\text{stuff}/\sqrt{n}\, p^{3/2} q^{3/2})}{n}\right]^n$$

$$= \left[1 + \frac{t^2}{\sigma_n^2}\frac{pq}{2} + \frac{d(n)}{n}\right]^n$$

$$= \left[1 + \frac{t^2\, \cancel{pq}}{n\cancel{pq}\; 2} + \frac{d(n)}{n}\right]^n$$

$$= e^{t^2/2}$$

$$Z_n \xrightarrow{d} N(0,1)$$

Note: Since $Z_n \dot\sim N(0,1)$

This leads to the approx

$$Y_n \dot\sim N(np, npq)$$

This works best when the Binomial is symmetric (p is near .5)

Ex: Prob a basketball player hits at least 9 free throws in 20. Suppose $p = .5$

Exact $P(Y_{20} \geq 9) = \sum_{y=9}^{20} \binom{20}{y} .5^{20} = .7483$

Est: $P(Y_{20} \geq 9) = 1 - P[Y_{20} \leq 8] = 1 - \Phi\left(\frac{8-10}{\sqrt{5}}\right)$

$= 1 - \Phi^{-1}(-.89) = .8133$

7.4 Continued

Ex.: Prob a basketball player hits at least 9 of 20 free throws (suppose $p = .5$)

Exact

$$P(Y_{20} \geq 9) = \sum_{y=9}^{20} \binom{20}{y} .5^{20} = \boxed{.7483}$$

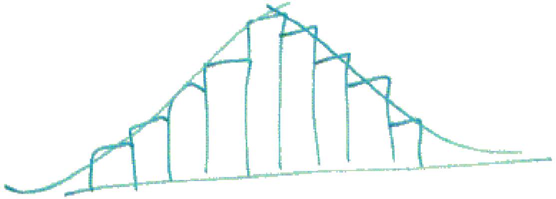
Est. $Y_n \sim N(\overset{=10}{np}, \overset{=5}{npq})$

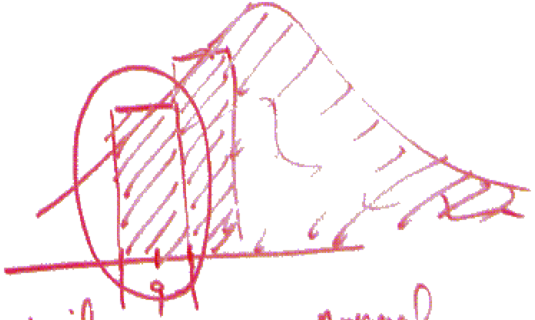
$$P(Y_n \geq 9) = 1 - P(Y_n \leq 8) = 1 - \Phi\left(\frac{8-10}{\sqrt{5}}\right)$$

$$= 1 - \Phi(-.89) = \boxed{.8133}$$

These aren't all that close.

We're to estimate a discrete dist with a continuous one.





binom'l 8.5 9.5 normal

$$P(Y_n \geq 9) = P(Y_n \geq 8.5)$$

$$= 1 - P(Y_n \leq 8.5) = 1 - \Phi\left(\frac{8.5-10}{\sqrt{5}}\right)$$

$$= \boxed{.7486}$$