7.3 Central Limit Thm

Thm 7.3.1 Let $Y_1, Y_2, \cdots$ be a sequence of R.V.

with CDFs

$G_1(y), G_2(y), \dots$ and

with MGFs

$M_1(t), M_2(t), \dots$

If $M(t)$ is the mgf of a CDF $G(y)$, and

If

$$\lim_{n\to\infty} M_n(t) = M(t) \quad \text{for all } t \in (-h,h)$$

then

$$\lim_{n\to\infty} G_n(y) = G(y) \quad \text{for all points of continuity of } G(y).$$

Preliminary:

$$\lim_{n\to\infty}\left(1+\frac{c}{n}\right)^{nb} = e^{cb}$$

$$\lim_{n\to\infty}\left(1+\frac{c}{n}+\frac{d(n)}{n}\right)^{nb} = e^{cb}$$

if $d(n)\to 0$ as $n\to\infty$

Ex: Let $X_1, X_2, \cdots, X_n$ be a RS from

$X_i \sim BIN(1,p)$

Consider $Y_n = \sum_{i=1}^{n} X_i \sim BIN(n,p)$

Let $p\to 0$ as $n\to\infty$ such that

$\mu = np$ remains fixed $\mu > 0$.

Then $M_n(t) = (pe^t+q)^n = \left(\frac{\mu e^t}{n} + 1 - \frac{\mu}{n}\right)^n$

$$= \left[1 + \frac{\mu(e^t-1)}{n}\right]^n = e^{\mu(e^t-1)}$$

$Y_n \xrightarrow{d} Y \sim POI(\mu)$

$\boxed{\text{Thm 7.3.2}}$ Central Limit Thm (CLT)

If $X_1, X_2, \ldots, X_n$ is a RS from a dist with mean μ and variance $\sigma^2 < \infty$, then the limiting dist of

$$Z_n = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{\sum X_i - n\mu}{\sqrt{n}\,\sigma} \qquad \bar{X} = \frac{1}{n}\sum X_i$$

is the standard normal, $Z_n \xrightarrow{d} Z \sim N(0,1)$ as $n \to \infty$.

Proof: The resnet hold for any dist with finite mean & variance, but the proof is easior by assuming the MGF exists! (Use the characteristic function otherwise (Fouier Transform))

Let $m(t)$ denote the MGF of $X-\mu$.

Then

$$m(t) = M_{X-\mu}(t)$$

$$m(0) = 1$$

$$m'(0) = E(X-\mu) = E(X) - \mu = \mu - \mu = 0$$

$$m''(0) = E(X-\mu)^2 = \sigma^2$$

So, expanding $m(t)$ in a Taylor Series about 0 gives.

$$m(t) = m(0) + m'(0)\,t + \frac{m''(\xi)}{2} t^2$$

$\xi \in (0,t)$

Hence, $m(t) = 1 + \frac{m''(\xi)t^2}{2}$

Now, the R.V. Z_n can be written as.

$$Z_n = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} = \frac{\sqrt{n}\left(\frac{1}{n}\sum X_i - \mu\right)}{\sigma} = \frac{\sum(X_i-\mu)}{\sqrt{n}\,\sigma}$$

So the MGF of Z_n is

$$M_{Z_n}(t) = E\left[\exp\left\{\frac{\sum(X_i-\mu)}{\sigma\sqrt{n}}t\right\}\right]$$

$$= M_{\sum(X_i-\mu)}\left(\frac{t}{\sigma\sqrt{n}}\right)$$

$$= \prod_{i=1}^{n} M_{X_i-\mu}\left(\frac{t}{\sigma\sqrt{n}}\right)$$

$$= \left[M_{X-\mu}\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n$$

$$= \left[m\left(\frac{t}{\sigma\sqrt{n}}\right)\right]^n$$

$$= \left[1 + \frac{m''(\xi)t^2}{2n\sigma^2}\right]^n, \quad 0 < |\xi| < \frac{|t|}{\sigma\sqrt{n}}$$

Now, since $\xi \in \left(\frac{-t}{\sigma\sqrt{n}}, \frac{t}{\sigma\sqrt{n}}\right)$, the $\xi \to 0$ as $n \to \infty$, Hence, $\lim_{n\to\infty} m''(\xi) = m''(0) = \sigma^2$

It follows that

$$= \left[1 + \frac{\sigma^2 t^2}{2n\sigma^2} - \frac{\sigma^2 t^2}{2n\sigma^2} + \frac{m''(\xi)t^2}{2n\sigma^2}\right]^n$$

$$= \Bigg[1 + \underbrace{\frac{\cancel{\sigma^2}t^2}{2n\cancel{\sigma^2}}}_{c = \frac{t^2}{2}} + \underbrace{\frac{(m''(\xi)-\sigma^2)t^2}{2n\sigma^2}}_{\frac{d(n)}{n} \to 0}\Bigg]^n = e^{\frac{t^2}{2}}$$

$$Z_n \xrightarrow{d} Z \sim N(0,1)$$

Ex: Roll 20 dice (6 sided die)

$$X_i \sim DU(6) \longrightarrow E(X_i) = \frac{6+1}{2} = \frac{7}{2}$$

Find $P[\sum X_i > 85]$

$$V(X_i) = \frac{6^2-1}{12} = \frac{35}{12}$$

$$Z_n = \frac{\sum X_i - n\mu}{\sqrt{n}\,\sigma} \xrightarrow{d} Z \sim N(0,1)$$

$$P\left[\sum X_i > 85\right] \approx P\left[\underbrace{\frac{\sum X_i - 20\left(\frac{7}{2}\right)}{\sqrt{20}\sqrt{\frac{35}{12}}}}_{Z} > \underbrace{\frac{85-20\left(\frac{7}{2}\right)}{\sqrt{20}\sqrt{\frac{35}{12}}}}\right]$$

$$= P[Z > 1.96] = .0250$$