

6.3 Joint transformations.

Thm 6.3.5

If X is a vector of discrete R.V. with joint pdf $f_X(x)$ and $Y=U(x)$ defines a one to one transformation, then joint pdf of Y is

$$f_Y(y) = f_X(x), \text{ when}$$

$$y = (y_1, y_2, \dots, y_k), x = (x_1, x_2, \dots, x_k)$$

and x is the solution of the transformation,

$$y = U(x), \quad (x \text{ depends on } y.)$$

If it is not 1-1, split it up over intervals where it is.

Then the equation $y=U(x)$ has a unique sol'n $x=x_j$ or $y_j=(x_{1j}, x_{2j}, \dots, x_{kj})$ over A_j . Then the pdf is

$$f_Y(y) = \sum_j f_X(x_j)$$

joint transforms of continuous R.V.'s can be accomplished, but the Jacobian has to be generalized. Suppose $u_1(x_1, x_2)$ and $u_2(x_1, x_2)$ are functions and x_1 & x_2 are unique solutions to $y_1 = u_1(x_1, x_2)$
 $y_2 = u_2(x_1, x_2)$.

Then the Jacobian of the transformation is

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$$

Ex: Suppose we want to transform x_1, x_2
into x_1 & $x_1 x_2$. Then

$$y_1 = x_1 \Rightarrow x_1 = y_1$$
$$y_2 = x_1 x_2 \Rightarrow x_2 = \frac{y_2}{x_1} = \frac{y_2}{y_1}$$

Note:

$$Y = u(\underbrace{x_1, x_2}_x), \text{ where } u(x_1, x_2) = (x_1, x_1 x_2)$$

$$x = u^{-1}(Y) = u^{-1}(Y_1, Y_2) = (Y_1, Y_2/Y_1)$$

$$S_0 \quad J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -\frac{y_2}{y_1^2} & \frac{1}{y_1} \end{vmatrix} = \frac{1}{y_1}$$

For a general transform $y = u(x)$, that has
a unique solution

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \dots & \dots & \frac{\partial x_1}{\partial y_k} \\ \vdots & \frac{\partial x_2}{\partial y_1} & \dots & \vdots \\ \vdots & \dots & \dots & \vdots \\ \frac{\partial x_k}{\partial y_1} & \dots & \dots & \frac{\partial x_k}{\partial y_k} \end{vmatrix}$$

EX: $X_1 \sim \text{EXP}(1)$ $X_1 \perp\!\!\!\perp X_2$
 $X_2 \sim \text{EXP}(1)$

$$f(x_1, x_2) = e^{-(x_1 + x_2)}$$

Transform from (x_1, x_2) to $(x_1 - x_2, x_1 + x_2)$

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} X_1 - X_2 \\ X_1 + X_2 \end{pmatrix} = u(x) = u \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\begin{aligned} y_1 &= x_1 - x_2 \\ y_2 &= x_1 + x_2 \end{aligned}$$

$$y_1 + y_2 = 2x_1$$

$$x_1 = \frac{y_1 + y_2}{2}$$

$$-y_1 = -x_1 + x_2$$

$$y_2 = x_1 + x_2$$

$$y_2 - y_1 = 2x_2$$

$$x_2 = \frac{y_2 - y_1}{2}$$

$$\text{so } x = u^{-1}(y) = \begin{pmatrix} \frac{y_1 + y_2}{2} \\ \frac{y_2 - y_1}{2} \end{pmatrix}$$

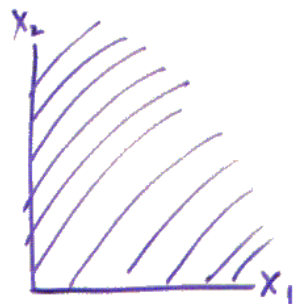
$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$f_Y(y) = f_X(x) |J|$$

$$= f_X \left(\frac{y_1 + y_2}{2}, \frac{y_2 - y_1}{2} \right) \cdot \frac{1}{2}$$

$$= e^{-\left(\frac{y_1 + y_2}{2} + \frac{y_2 - y_1}{2} \right)} \cdot \frac{1}{2}$$

$$= \frac{1}{2} e^{-y_2}, \quad y \in B \Rightarrow \text{what is } B?$$



$$A = \{ (x_1, x_2) \mid f(x) > 0 \}$$

x_1, x_2

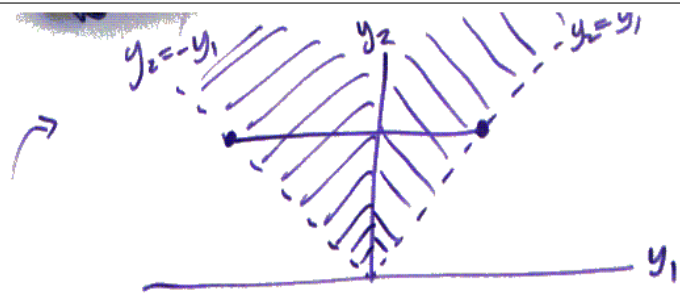
$$x_1 > 0, x_2 > 0.$$

The transform $x_1 = \frac{y_1 + y_2}{2} > 0$

$$2x_1 = y_1 + y_2 > 0 \Rightarrow y_1 > -y_2$$

$$x_2 = \frac{y_2 - y_1}{2} > 0$$

$$\Rightarrow 2x_2 = y_2 - y_1 > 0 \Rightarrow y_2 > y_1$$



So $y_2 > 0$ and $-y_2 < y_1 < y_2$
 $|y_1| < y_2 < \infty$

$$f_Y(y) = \frac{1}{2} e^{-y_2} I_{(0, \infty)}(y_2) I_{(y_1, \infty)}(y_1)$$

The marginal for Y_1 is:

$$\begin{aligned} f_{Y_1}(y_1) &= \int_{-\infty}^{\infty} f_{Y_1, Y_2}(y_1, y_2) dy_2 \\ &= \int_{-\infty}^{\infty} \frac{1}{2} e^{-y_2} I_{(|y_1|, 1, \infty)}(y_2) I_{(0, \infty)}(y_2) dy_2 \\ &= \int_{|y_1|}^{\infty} \frac{1}{2} e^{-y_2} dy_2 = -\frac{1}{2} e^{-y_2} \Big|_{|y_1|}^{\infty} \\ &= \frac{1}{2} e^{-|y_1|} \end{aligned}$$

$$Y_1 \sim \text{DE}(1, 0)$$

The marginal for Y_2 is

$$\begin{aligned} f_{Y_2}(y_2) &= \int_{-\infty}^{\infty} f_Y(y) dy_1 \\ &= \int_{-\infty}^{\infty} \frac{1}{2} e^{-y_2} \underbrace{I_{(|y_1|, 1, \infty)}(y_2)}_{|y_1| < y_2 \Rightarrow \underbrace{-y_2 < y_1 < y_2}_{I_{(-y_2, y_2)}(y_1)}} I_{(0, \infty)}(y_2) dy_1 \\ &= \int_{-y_2}^{y_2} \frac{1}{2} e^{-y_2} I_{(0, \infty)}(y_2) dy_1 \\ &= \frac{1}{2} e^{-y_2} I_{(0, \infty)}(y_2) \int_{-y_2}^{y_2} dy_1 \\ &= \frac{1}{2} e^{-y_2} I_{(0, \infty)}(y_2) \cdot 2y_2 \\ &= y_2 e^{-y_2} I_{(0, \infty)}(y_2) \Rightarrow Y_2 \sim \text{GAM}(1, 2) \end{aligned}$$

EX: Let $X_i \sim \text{GAM}(\theta, k_i)$ $i=1, 2$

$$X_1 \perp\!\!\!\perp X_2$$

We want to find the dist of

$$\frac{X_1}{X_1+X_2} \quad \text{and} \quad \frac{X_2}{X_1+X_2}$$

do later.

$$Z_1 = \frac{X_1}{X_1+X_2}$$

$$Z_2 = X_2$$

$$X_1 = \frac{Z_1 Z_2}{1-Z_1}$$

$$X_2 = Z_2$$

$$J = \begin{vmatrix} \frac{\partial X_1}{\partial Z_1} & \frac{\partial X_1}{\partial Z_2} \\ \frac{\partial X_2}{\partial Z_1} & \frac{\partial X_2}{\partial Z_2} \end{vmatrix} = \begin{vmatrix} \frac{Z_2}{(1-Z_1)^2} & \frac{Z_1}{1-Z_1} \\ 0 & 1 \end{vmatrix} = \frac{Z_2}{(1-Z_1)^2}$$

The pdf of X_i is

$$f_{X_i}(x_i) = \frac{1}{\Gamma(k_i) \theta^{k_i}} x_i^{k_i-1} e^{-\frac{x_i}{\theta}} I_{(0,\infty)}(x_i)$$

The joint pdf of X_1 & X_2 is

$$\begin{aligned} f_{X_1, X_2}(x_1, x_2) &= f_1(x_1) \cdot f_2(x_2) \\ &= \frac{1}{\Gamma(k_1) \Gamma(k_2) \theta^{k_1+k_2}} x_1^{k_1-1} x_2^{k_2-1} e^{-\frac{x_1+x_2}{\theta}} I_{(0,\infty)}(x_1) \times I_{(0,\infty)}(x_2) \end{aligned}$$

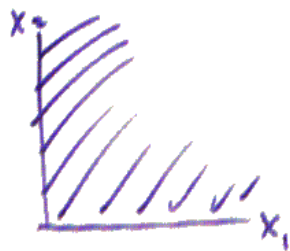
So

$$f_z(z) = f_X(x) |J|$$

$$= f_X\left(\frac{z_1 z_2}{1-z_1}, z_2\right) \frac{z_2}{(1-z_1)^2}$$

$$= \frac{1}{\Gamma(k_1)\Gamma(k_2)\theta^{k_1+k_2}} \left(\frac{z_1 z_2}{1-z_1}\right)^{k_1-1} z_2^{k_2-1} e^{-\frac{1}{\theta}\left(\frac{z_1 z_2}{1-z_1} + z_2\right)} \frac{z_2}{(1-z_1)^2} I_{(0,\infty)}\left(\frac{z_1 z_2}{1-z_1}\right) I_{(0,\infty)}(z_2)$$

$$= \frac{1}{\Gamma(k_1)\Gamma(k_2)\theta^{k_1+k_2}} \frac{z_1^{k_1-1} z_2^{k_1+k_2-1}}{(1-z_1)^{k_1+1}} e^{-\frac{1}{\theta}\left(\frac{z_1 z_2 + z_2(1-z_1)}{1-z_1}\right)} I_{(0,\infty)}\left(\frac{z_1 z_2}{1-z_1}\right) I_{(0,\infty)}(z_2)$$



Note that $0 < \frac{z_1 z_2}{1-z_1} < \infty$

$0 < z_1 < 1$ $0 < z_2 < \infty$

$I_{(0,1)}(z_1)$



We want the dist of z_1

$$f(z_1) = \int_{-\infty}^{\infty} f_2(z) dz_2$$

$$= \frac{I_{(0,1)}(z_1) z_1^{k_1-1}}{\underbrace{\Gamma(k_1) \Gamma(k_2) \theta^{k_1+k_2} (1-z_1)^{k_1+1}}_{c(z_1)}}$$

$$\int_0^{\infty} z_2^{k_1+k_2-1} e^{-\frac{z_2}{\theta(1-z_1)}} dz_2$$

$$u = \frac{z_2}{\theta(1-z_1)} \Rightarrow z_2 = u \theta(1-z_1)$$

$$dz_2 = \theta(1-z_1) du$$

$$= c(z_1) \int_0^{\infty} [u \theta(1-z_1)]^{k_1+k_2-1} e^{-u} \theta(1-z_1) du$$

$$= c(z_1) \theta^{k_1+k_2} (1-z_1)^{k_1+k_2} \int_0^{\infty} u^{k_1+k_2-1} e^{-u} du$$

$$= \frac{I_{(0,1)}(z_1) z_1^{k_1-1} \Gamma(k_1+k_2) \theta^{k_1+k_2} (1-z_1)^{k_1+k_2}}{\Gamma(k_1) \Gamma(k_2) \theta^{k_1+k_2} (1-z_1)^{k_1+1}}$$

$$= \frac{\Gamma(k_1+k_2)}{\Gamma(k_1) \Gamma(k_2)} z_1^{k_1-1} (1-z_1)^{k_2-1} \frac{I_{(0,1)}(z_1)}{z_1} \\ z_1 \sim \text{BETA}(k_1, k_2)$$

$$B(k_1, k_2) = \frac{\Gamma(k_1) \Gamma(k_2)}{\Gamma(k_1+k_2)} \\ = \int_0^1 z_1^{k_1-1} (1-z_1)^{k_2-1} dz_1$$

Try and prove it! If you do, I'll add 5% to any one of your tests.