

Chap 6

Functions of Random Vars.

Suppose X represents the age in weeks of some component. Perhaps age in days is desired instead.

$$W = \ln X$$

$$\bar{X} = \frac{1}{n} X_1 + \frac{1}{n} X_2 + \dots + \frac{1}{n} X_n$$

We will discuss methods of deriving the pdf of the new random variable that is a function of the other random vars.

6.2 The CDF Technique.

Assume X has CDF $F_X(x)$ and we want to find the CDF of $Y = u(X)$

Idea: Express the CDF of Y in terms of the dist of X .

Define $A_y = \{x | u(x) \leq y\}$. It follows that $[Y \geq y]$ and $[x \in A_y]$ are equivalent sets and as such

$$F_Y(y) = P[u(X) \leq y] = P[x \in A_y]$$

often $x \in A_y$ can be expressed as $x_1 \leq x \leq x_2$ where the limits depend on y

In the cont. case, if

$u(x) \leq y$ is equiv. to $x_1 \leq x \leq x_2$

$$F_Y(y) = P[u(x) \leq y] = P[x_1 \leq x \leq x_2]$$

$$= \int_{x_1}^{x_2} f_X(x) dx = F_X(x_2) - F_X(x_1)$$

and the pdf is

$$\frac{dF_Y(y)}{dy} = f_Y(y)$$

Ex: Suppose that the CDF is

$$F(x) = 1 - e^{-2x}, \quad 0 < x < \infty$$

Consider $Y = e^X$. The CDF of Y is therefore

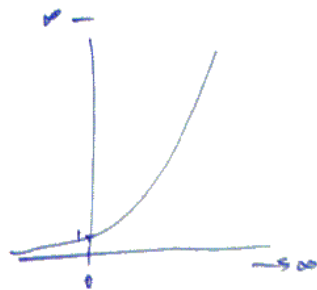
$$F_Y(y) = P[Y \leq y] = P[e^X \leq y]$$

$$= P[X \leq \ln y] = F_X(\ln y)$$

$$= 1 - e^{-2 \ln y} = 1 - e^{\ln y^{-2}}$$

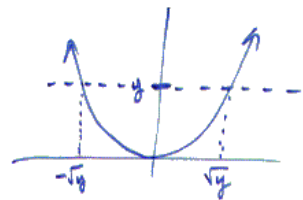
$$= 1 - y^{-2}$$

Thus, the pdf of Y is $f_Y(y) = \frac{2}{y^3}, \quad 1 < y < \infty$



Ex: Consider a cont. R.V. X .

Let $Y = X^2$. then



$$F_Y(y) = P[X^2 \leq y] = P[|X| \leq \sqrt{y}]$$

$$= P[-\sqrt{y} \leq X \leq \sqrt{y}] = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$

The pdf of Y is

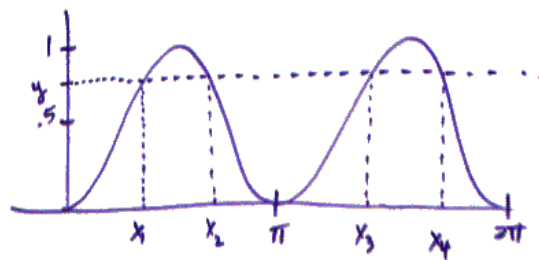
$$f(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} F_X(\sqrt{y}) - \frac{d}{dy} F_X(-\sqrt{y})$$

$$= f_X(\sqrt{y}) \left(\frac{1}{2\sqrt{y}} \right) - f_X(-\sqrt{y}) \left(\frac{-1}{2\sqrt{y}} \right)$$

$$= \frac{1}{2\sqrt{y}} \left[f_X(\sqrt{y}) + f_X(-\sqrt{y}) \right], y \geq 0$$

Ex: Suppose $X \sim \text{UNIF}(0, 2\pi)$

Consider $Y = \sin^2 X$



$$F_Y(y) = P[Y \leq y] = P(X \leq x_1) + P(x_2 \leq X \leq x_3) + P(X \geq x_4)$$

$$= \underbrace{P(X \leq x_1) + P(x_2 \leq X \leq \pi)}_{\text{same}} + \underbrace{P(\pi \leq X \leq x_3) + P(X \geq x_4)}_{\text{same}}$$

$$= 2P(X \leq x_1) + 2P(x_2 \leq X \leq \pi)$$

x_1, x_2 are the solutions to $y = \sin^2 x$

EX. 6.2.3.

$$\Theta \sim \text{UNIF}(0, 2\pi)$$

$$f(\theta) = \frac{1}{2\pi}, \quad 0 \leq \theta \leq 2\pi$$

$$F_{\Theta}(\theta) = \int_0^{\theta} \frac{1}{2\pi} d\theta = \frac{\theta}{2\pi}$$

$$Y = \tan \theta.$$

$$F_Y(y) = P\{\tan \theta \leq y\}$$

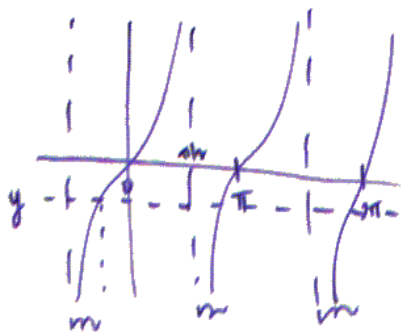
$$= P\left\{\frac{\pi}{2} \leq \theta < \pi + \tan^{-1} y\right\} + P\left\{\frac{3\pi}{2} \leq \theta \leq 2\pi + \tan^{-1} y\right\}$$

$$= F_{\Theta}(\pi + \tan^{-1} y) - F_{\Theta}\left(\frac{\pi}{2}\right) + F_{\Theta}(2\pi + \tan^{-1} y) - F_{\Theta}\left(\frac{3\pi}{2}\right)$$

$$= \frac{1}{2\pi} \left[\pi + \tan^{-1} y - \frac{\pi}{2} \right] + \frac{1}{2\pi} \left[2\pi + \tan^{-1} y - \frac{3\pi}{2} \right]$$

$$= \frac{1}{2\pi} \left[\frac{\pi}{2} + \tan^{-1} y \right] + \frac{1}{2\pi} \left[\frac{\pi}{2} + \tan^{-1} y \right] = \frac{1}{\pi} \left[\frac{\pi}{2} + \tan^{-1} y \right]$$

$$\text{the pdf is } f_Y(y) = \frac{d}{dy} \left(\frac{1}{\pi} \left[\frac{\pi}{2} + \tan^{-1} y \right] \right) = \frac{1}{\pi} \frac{1}{1+y^2} \sim \text{CAUCHY}(0, 1)$$



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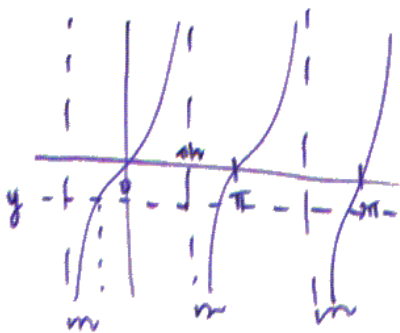
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Thm 6.2.1

Let $X = (X_1, \dots, X_k)$ be a k -dimensional r.v. with joint pdf $f(x_1, x_2, \dots, x_k)$.

If $Y = U(X)$ is a function of X , then

$$F_Y(y) = P[u(X) \leq y] = \int \dots \int_{A_y} f(x_1, \dots, x_k) dx_1 \dots dx_k$$

where $A_y = \{x \mid u(x) \leq y\}$
↑ bold face

EX: $X_1 \sim \text{EXP}(1)$
 $X_2 \sim \text{EXP}$
 $X_1 \perp\!\!\!\perp X_2$

Find the dist of $Y = X_1 + X_2$

$$A_y = \{ (x_1, x_2) \mid 0 \leq x_1 + x_2 \leq y \}$$

$0 \leq x_1 \leq y - x_2$ and
 $0 \leq x_2 \leq y$

$$F_Y(y) = \int_0^y \int_0^{y-x_2} e^{-(x_1+x_2)} dx_1 dx_2$$

$$= 1 - e^{-y} - ye^{-y}$$

$$f_Y(y) = ye^{-y}, y > 0$$

$$Y \sim \text{GAM}(1, 2)$$

