$\boxed{5.4}$ Cont

extension of thm 5.4.1

$\boxed{\text{Thm } 5.4.4}$

If X and Y are jointly dist R.V.

h(x,y) is a function, then

$$E[h(X,Y)] = E_X\left[E[h(X,Y)|X]\right]$$

This says we can find the expectation of h(X,Y) by first finding

$$E\left[h(x,Y)\,\middle|\,x\right]$$ and then taking

the expectation wrt X.

<u>Thm 5.4.5</u> If X & Y are jointly dist R.V.'s, and g(x) is a function, then

$$E\left[g(X)Y\,\middle|\,x\right] = g(x)E\left[Y\,\middle|\,x\right]$$

<u>Corollary</u>:

$$E\left\{E\left(g(X)Y\,\middle|\,X\right)\right\} = E\left[g(X)E[Y|X]\right]$$

<u>Ex</u>: 5.4.3

$(X,Y) \sim MULT(n, p_1, p_2)$, then

$X \sim BIN(n, p_1)$

$Y \sim BIN(n, p_2)$

$Y|x \sim BIN(n-x, p)$, $p = \frac{p_2}{1-p_1}$

$Var(X) = E(X^2) - \mu^2$

$E(X) = np_1$

$E(Y) = np_2$

$E[Y|x] = \frac{(n-x)p_2}{1-p_1}$

Find the covariance of X & Y.

$$E[XY] = E\big[E[XY|X]\big] \qquad \text{Thm 5.4.1}$$

$$= E\big[X\,E[Y|X]\big] \qquad \text{Thm 5.4.5 (Corollary)}$$

$$= E\left[\frac{X(n-X)p_2}{1-p_1}\right]$$

$$= \frac{p_2}{1-p_1}E\left[nX - X^2\right]$$

$$= \frac{p_2}{1-p_1}\Big[nE(X) - \underbrace{E(X^2)}_{V(X)+\mu^2}\Big]$$

$$= \frac{p_2}{1-p_1}\left[n^2p_1 - np_1(1-p_1) - n^2p_1^2\right]$$

$$= \frac{np_1p_2}{1-p_1}\Big[n-1+\underbrace{p_1 - np_1}_{-p_1(n-1)}\Big]$$

$$= \frac{n(n-1)p_1p_2}{\cancel{1-p_1}}\left[\cancel{1-p_1}\right]$$

$$E[XY] = n(n-1)p_1p_2$$

Thus, $\mathrm{Cov}(X,Y) = E(XY) - E(X)E(Y)$

$$= n(n-1)p_1p_2 - n^2p_1p_2$$

$$= n^2p_1p_2 - np_1p_2 - n^2p_1p_2 = \boxed{-np_1p_2}$$

Thm 5.4.6

If $E(Y|x)$ is a linear function of x, then

$$E(Y|x) = \mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x-\mu_1)$$

$$\mu_1 = E(X),\ \mu_2 = E(Y),\ \sigma_1^2 = V(X),\ \sigma_2^2 = V(Y)$$

and $$E_X[V(Y|X)] = \sigma_2^2(1-\rho^2)$$

Bivariate Normal Dist

A pair of cont. R.V X & Y is said to have a bivariate normal dist if it has the joint pdf:

$$f(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]\right\}$$

$$(X,Y) \sim BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$$

$-\infty < \mu_1 < \infty$ $\quad \sigma_1 > 0$ $\quad -1 < \rho < 1$

$-\infty < \mu_2 < \infty$ $\quad \sigma_2 > 0$

Thm 5.4.7

If $(X,Y) \sim BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$, then

$X \sim N(\mu_1, \sigma_1^2)$ $\quad Y \sim N(\mu_2, \sigma_2^2)$ and

ρ is the correlation coeff between them.

Note: we learned that if $X \perp\!\!\!\perp Y$, then $Cov(X,Y) = 0$
$\rho = 0$

But here in Thm 5.4.7, we learn that if $\rho = 0$, then
the joint pdf (X,Y) can be factored into a product of marginals.

Hence, for the NORMAL dist, independence & uncorrelated are the same! This isn't true in general!

$\boxed{\text{Thm } 5.4.8}$ If $(X,Y) \sim BVN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho)$, then

1. $Y|x \sim N\left[\mu_2 + \rho \frac{\sigma_2}{\sigma_1}(x-\mu_1), \sigma_2^2(1-\rho^2)\right]$

2. $X|y \sim N\left[\mu_1 + \rho \frac{\sigma_1}{\sigma_2}(y-\mu_2), \sigma_1^2(1-\rho^2)\right]$

5.5 Joint MGF

DEF: the joint MGF of $X=(X_1,\dots,X_k)$ if it exists, is defined as

$$M_X(t) = E\left[\exp\left(\sum_{i=1}^{k} t_i X_i\right)\right], \text{ where } t=(t_1,\dots,t_k)$$

and $-h<t_i<h$ for some $h>0$.

$\boxed{\text{Thm 5.5.1}}$ If $M_{X,Y}(t_1,t_2)$ exists, then RVs are independent iff

$$M_{X,Y}(t_1,t_2) = M_X(t_1)\,M_Y(t_2)$$

Mixed Moments

$$E\left[X_i^r X_j^s\right] = \frac{\partial^r}{\partial t_i^r}\frac{\partial^s}{\partial t_j^s} M_X(t)\Big|_{t_m=0}$$

$m=1,\dots,k$

Marginal dist MGF's

$$M_X(t_1) = M_{X,Y}(t_1,0)$$

$$M_Y(t_2) = M_{X,Y}(0,t_2)$$

Ex: $X = (X_1, \cdots, X_k) \sim MULT(n, p_1, \cdots, p_k)$

Marginals $X_i \sim BIN(n, p_i)$

The joint MGF

$$M_X(t) = E\left[\exp\left(\sum t_i X_i\right)\right]$$

$$= \sum \cdots \sum \frac{n}{x_1! \cdots x_{k+1}!} (p_1 e^{t_1})^{x_1} \cdots (p_k e^{t_k})^{x_k} p_{k+1}^{x_{k+1}}$$

$$= \left(p_1 e^{t_1} + \cdots + p_k e^{t_k} + p_{k+1}\right)^n$$

$p_{k+1} = 1 - \sum p_i$

$x_{k+1} = n - \sum x_i$

Suppose

$(X_1, X_2, X_3) \sim MULT(n, p_1, p_2, p_3)$

$M_{X_1,X_2,X_3}(t_1,t_2,t_3) =$

$$(p_1 e^{t_1} + p_2 e^{t_2} + p_3 e^{t_3} + p_4)^n$$

$p_4 = 1 - p_1 - p_2 - p_3$

$M_{X_1,X_2}(t_1,t_2) = M_{X_1,X_2,X_3}(t_1,t_2,0)$

$$= (p_1 e^{t_1} + p_2 e^{t_2} + \cancel{p_3} + 1 - p_1 - p_2 - \cancel{p_3})^n$$

$(X_1, X_2) \sim MULT(n, p_1, p_2)$

Ex: $X = (X_1, \cdots, X_k) \sim MULT(n, p_1, \cdots, p_k)$

Marginals $X_i \sim BIN(n, p_i)$

The joint MGF

$$M_X(t) = E\left[\exp\left(\sum t_i X_i\right)\right]$$

$$= \sum \cdots \sum \frac{n}{x_1! \cdots x_{k+1}!} (p_1 e^{t_1})^{x_1} \cdots (p_k e^{t_k})^{x_k} p_{k+1}^{x_{k+1}}$$

$$= (p_1 e^{t_1} + \cdots + p_k e^{t_k} + p_{k+1})^n$$

$p_{k+1} = 1 - \sum p_i$

$x_{k+1} = n - \sum x_i$

Suppose

$(X_1, X_2, X_3) \sim MULT(n, p_1, p_2, p_3)$

$M_{X_1, X_2, X_3}(t_1, t_2, t_3) =$

$$(p_1 e^{t_1} + p_2 e^{t_2} + p_3 e^{t_3} + \underbrace{p_4}_{1-p_1-p_2-p_3})^n$$

$$M_{X_1, X_2}(t_1, t_2) = M_{X_1, X_2, X_3}(t_1, t_2, 0)$$

$$= (p_1 e^{t_1} + p_2 e^{t_2} + \cancel{p_3} + 1 - p_1 - p_2 - \cancel{p_3})^n$$

$(X_1, X_2) \sim MULT(n, p_1, p_2)$