$\boxed{\text{Thm } 5.3.1}$

If ρ is the correlation coeff of $X \& Y$,

then $-1 \le \rho \le 1$

and $\rho = \pm 1$ iff $Y = aX + b$ with prob 1.

proof:

$$h(t) = E\left[(X-\mu_x)t + (Y-\mu_y)\right]^2$$

$$= E\left[(X-\mu_x)^2 t^2 + 2t(X-\mu_x)(Y-\mu_y) + (Y-\mu_y)^2\right]$$

$$= t^2 E(X-\mu_x)^2 + 2tE\left[(X-\mu_x)(Y-\mu_y)\right] + E(Y-\mu_y)^2$$

Since $h(t) \ge 0$ for all values of t (why?)
$h(t)$ is a quadratic function, $h(t)$ has at most one real root and thus must have a non positive discriminant.

$$b^2 - 4ac \le 0$$

$$\left[2\underbrace{E\left[(X-\mu_X)(Y-\mu_Y)\right]}_{\sigma_{XY}}\right]^2 - 4\underbrace{E(X-\mu_X)^2}_{\substack{V(X)\\ \sigma_X^2}}\underbrace{E(Y-\mu_Y)^2}_{\substack{V(Y)\\ \sigma_Y^2}} \le 0$$

$$4\sigma_{XY}^2 - 4\sigma_X^2\sigma_Y^2 \le 0$$

$$4\sigma_{XY}^2 \le 4\sigma_X^2\sigma_Y^2$$

$$|\sigma_{XY}| \le \sigma_X\sigma_Y$$

$$-\sigma_X\sigma_Y \le \sigma_{XY} \le \sigma_X\sigma_Y$$

$$-1 \le \frac{\sigma_{XY}}{\sigma_X\sigma_Y} \le 1$$

$$-1 \le \rho \le 1$$

5.4 Conditional Expectation.

DEF: Conditional Exp.

$$E[Y|x] = \sum_{\text{all } y} y\, f(y|x) \quad (X \,\&\, Y \text{ are discrete})$$

$$= \int_{-\infty}^{\infty} y\, f(y|x) \quad (x \,\&\, y \text{ are cont.})$$

other Notations

$$E[Y|X=x] \qquad E_{Y|X}(Y)$$

Note $E[Y|x]$ is a function of x!

$$g(x) = E[Y|x]$$

$\boxed{\text{Thm 5.4.1}}$ If $X \,\&\, Y$ are jointly dist R.V.'s

then $E_x[E(Y|X)] = E(Y)$

$g(x) = E(Y|x) =$ function of X.

$$E_x(g(x)) = \sum g(x)\, f(x)$$

$$\int_{-\infty}^{\infty} g(x)\, f(x)\, dx$$

$E(E(Y|X))$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f(y|x)\, f(x) dx$$

$$= \int_{-\infty}^{\infty} y\, f(y)\, dy$$

proof.

$$E[E(Y|X)] = \int_{-\infty}^{\infty} E(Y|x)\, f_1(x)\, dx$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} y\, f(y|x)\, dy \right] f_1(x)\, dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y\, f(y|x)\, f_1(x)\, dy\, dx$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y\, f(x,y)\, dy\, dx$$

$$= \int_{-\infty}^{\infty} y \int_{-\infty}^{\infty} f(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} y\, f_2(y)\, dy$$

$$= E(Y)$$

Ex: 5.4.2

$X \sim POI(20)$ $\quad E(X) = \mu = 20$

$Y|X \sim BIN(X, .85)$

$E(Y) \Rightarrow$ 2 ways to find this.

1) find pdf of Y and then compute the mean
2) $E(E(Y|X)) = E(Y)$

$$E(Y) = E(E(Y|X)) = E(X(.85)) = 85\, E(X)$$

$$= 20(85) = 17$$

$\boxed{\text{Thm 5.4.2}}$ If X&Y are independent,

$E(Y|x) = E(Y)$

$E(X|y) = E(X)$

Def: The conditional variance of Y given X=x is

$$V(Y|x) = E\{[Y - E(Y|X)]^2 \mid X\}$$
$$= E[Y^2|x] - [E(Y|x)]^2$$

$\boxed{\text{Thm 5.4.3}}$ If X&Y are jointly dist, then

$$Var(Y) = E_X[Var(Y|X)] + Var_X[E(Y|X)]$$

Ex:

$$Var(Y) = E(Var(Y|X)) + Var(E(Y|X))$$
$$= E(X(.85)(.15)) + Var(X(.85))$$
$$= .85(.15)\,E(X) + (.85)^2\,Var(X)$$
$$= .85(.15)(20) + (.85)^2(20)$$
$$= 2.55 + 14.45$$
$$= 17$$