5.2 Properties of Expected Values.

Thm. If $X = (X_1, \cdots, X_k)$ has joint pdf $f(x_1, x_2, \cdots, x_k)$ and if $Y = u(X_1, \cdots, X_k)$ is a function of X, then

$$E(Y) = E_X[u(x_1, \cdots, x_k)], \text{ where}$$

$$E_X[u(x_1, \cdots, x_k)] = \sum_{x_1} \cdots \sum_{x_k} u(x_1, \cdots, x_k) f(x_1, \cdots, x_k), \text{ if } X \text{ is discrete}$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} u(x_1, \cdots, x_k) f(x_1, \cdots, x_k)\, dx_1 dx_2 \cdots dx_k,$$

if X is contin.

Thm 5.2.2

If X_1 & X_2 are R.V with joint pdf $f(x_1, x_2)$, then

$$E(X_1 + X_2) = E(X_1) + E(X_2)$$

proof. (Discrete case)

$$E(X_1 + X_2) = \sum_{x_1}\sum_{x_2} (x_1 + x_2) f(x_1, x_2)$$

$$= \sum_{x_1}\sum_{x_2} x_1 f(x_1, x_2) + \sum_{x_1}\sum_{x_2} x_2 f(x_1, x_2)$$

$$= \sum_{x_1} x_1 \sum_{x_2} f(x_1, x_2) + \sum_{x_2} x_2 \sum_{x_1} f(x_1, x_2)$$

$$= \sum_{x_1} x_1 f_1(x_1) + \sum_{x_2} x_2 f_2(x_2)$$

$$= E(X_1) + E(X_2)$$

Thm If X & Y are indep R.V. and $g(x)$ & $h(y)$ are functions, then

$$E[g(X)h(Y)] = E(g(X))E(h(Y))$$

proof. cont. case is in the book.

DEF. The covariance of a pair of random variables X & Y is defined by

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)] = \sigma_{XY}$$

Thm. If X & Y are RV, and a, b are const then

a) $Cov(aX, bY) = ab\, Cov(X,Y)$

b) $Cov(X+a, Y+b) = Cov(X,Y)$

c) $Cov(X, aX+b) = a\, Var(X)$

Thm 5.2.5 If X & Y are RV's, then

$$Cov(X,Y) = E(XY) - E(X)E(Y) \text{ and}$$

If X & Y are independent, then

$$Cov(X,Y) = 0.$$

Proof: $Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$

$$= E[XY - X\mu_Y - \mu_X Y + \mu_X\mu_Y]$$

$$= E(XY) - \mu_Y E(X) - \mu_X E(Y) + \mu_X\mu_Y$$

$$= E(XY) - \mu_X\mu_Y - \cancel{\mu_X\mu_Y} + \cancel{\mu_X\mu_Y}$$

$$= E(XY) - \mu_X\mu_Y$$

If X, Y are indep, then $Cov(X,Y) = E(X)E(Y) - \mu_X\mu_Y$

$$= \mu_X\mu_Y - \mu_X\mu_Y = 0$$

$\boxed{\text{Thm } 5.26}$

If X_1 & X_2 are RV w/pdf $f(x_1, x_2)$,

$$V(X_1 + X_2) = V(X_1) + V(X_2) + 2\,\text{Cov}(X_1, X_2)$$

If $X_1 \perp\!\!\!\perp X_2$ ($\perp\!\!\!\perp$ = is independent of), then

$$V(X_1 + X_2) = V(X_1) + V(X_2)$$

proof: DEF: $\mu_1 = E(X_1)$, $\mu_2 = E(X_2)$

$$V(X_1+X_2) = E(X_1+X_2)^2 - \left[E(X_1+X_2)\right]^2$$
$$= E\left[X_1^2 + 2X_1X_2 + X_2^2\right] - \left[\underbrace{E(X_1)}_{\mu_1} + \underbrace{E(X_2)}_{\mu_2}\right]^2$$
$$= E(X_1^2) + 2E(X_1X_2) + E(X_2^2) - \mu_1^2 - 2\mu_1\mu_2 - \mu_2^2$$
$$= \left[E(X_1^2) - \mu_1^2\right] + \left[E(X_2^2) - \mu_2^2\right] + 2\left[E(X_1X_2) - \mu_1\mu_2\right]$$
$$= V(X_1) + V(X_2) + 2\,\text{Cov}(X_1, X_2)$$

Note: If $X_1 \perp\!\!\!\perp X_2$, then

$$V(X_1 + X_2) = V(X_1) + V(X_2)$$

In general,

(a) $E\left[\sum_{i=1}^{k} a_i X_i\right] = \sum_{i=1}^{k} a_i E(X_i)$

(b) $V\left[\sum_{i=1}^{k} a_i X_i\right] = \sum_{i=1}^{k} a_i^2 V(X_i) + 2\sum\sum_{i<j} a_i a_j \text{Cov}(X_i, X_j)$

(c) If $X_1, X_2, \ldots, X_k$ are indep, then

$$V\left[\sum_{i=1}^{k} a_i X_i\right] = \sum_{i=1}^{k} a_i^2 V(X_i)$$

Ex: Suppose $Y \sim BIN(n,p)$

Because binomial R.V.'s result from n independent Bernoulli R.V.'s, then

$$Y = \sum_{i=1}^{n} X_i, \quad \text{where } X_i \sim BER(p)$$

$$E[Y] = E\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} p = np$$

$$V[Y] = V\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} V(X_i) = \sum_{i=1}^{n} pq = npq$$

$\boxed{5.3}$ Correlation

X \ Y	-1	0	1	
-1	0	1/4	0	1/4
0	1/4	0	1/4	1/2
1	0	1/4	0	1/4
	1/4	1/2	1/4	

Marginal of X & Y:

x	$f_1(x)$
-1	1/4
0	1/2
1	1/4

y	$f_2(y)$
-1	1/4
0	1/2
1	1/4

$$f(x,Y) = f_X(x)\, f_Y(Y)$$

$$f(1,1) = f_X(1)\, f_Y(1)$$

$$0 \neq \frac{1}{4}\cdot\frac{1}{4}$$

$$\Rightarrow X \not\perp\!\!\!\perp Y$$

$$E(Y) = E(X) = \sum_{\text{all } x} x f_1(x)$$

$$= (-1)\left(\tfrac{1}{4}\right) + 0\left(\tfrac{1}{2}\right) + 1\left(\tfrac{1}{4}\right)$$

$$= 0$$

$$E(XY) = \sum_{x}\sum_{Y} xY f(x,Y) = (-1)(-1)(0)$$

$$+ (-1)(0)\left(\tfrac{1}{4}\right) + (-1)(1)(0) + (0)(-1)\left(\tfrac{1}{4}\right)$$

$$+ (0)(0)(0) + 0\left(\tfrac{1}{4}\right)(1) + 1(-1)(0)$$

$$+ (1)\left(\tfrac{1}{4}\right)(0) + 1(1)(0) = 0$$

$$Cov(X,Y) = E(XY) - E(X)E(Y) = 0$$

Remember! If $X \perp\!\!\!\perp Y$, then $Cov(X,Y) = 0$

But it does not always work the other way around! (The converse is false)

DEF: If X & Y are random variables with variances σ_X^2 & σ_Y^2 and cov σ_{XY} then the correlation coefficient of X & Y is

$$\rho = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

If $\rho = 0$, then X & Y are said to be uncorrelated.

Thm: If ρ is the correlation coeff of X and Y, then $-1 \leq \rho \leq 1$

and $\rho = \pm 1$ iff $Y = aX + b$ with prob 1.

for some $a \neq 0$ and b.