4.4 Independent R.V.

Table:

$f(x_1,x_2)$	$x_2 = 0$	$x_2 = 1$	$x_2 = 2$	$f_1(x_1)$
$x_1 = 0$	.1	.2	.1	.4
$x_1 = 1$	.1	.2	.1	.4
$x_1 = 2$	.1	.1	0	.2
$f_2(x_2)$	.3	.5	.2	

This a pdf since all values are ≥ 0 and $.1+.2+.1+.1+.2+.1+.1+.1=1$

Note that $P(X_2=1, X_1=0) = P(X_2=1)P(X_1=0)$

$$f(0,1) = f_2(1)\, f_1(0)$$

$$.2 = .5\,(.4) \checkmark$$

Hence, the events $X_2=1$ & $X_1=0$ are independent events.

However, $X_1=0$ & $X_2=0$ are not independent since

$$f(0,0) \neq f_1(0)\cdot f_2(0)$$

$$.1 \neq .4\,(.3)$$

Two variables X_1 & X_2 are independent if

$$f(x_1,x_2) = f_1(x_1)\cdot f_2(x_2)$$

for all x_1 and x_2

For continuous dist, a similar concept applies

DEF 4.4.1

Independent RV's

Random Var $X_1, X_2, \cdot, X_k$ are said to be independent if for every $a_i < b_i$

$$P[a_1 \le X_1 \le b_1, a_2 \le X_2 \le b_2, \quad , a_k \le X_k \le b_k] = \prod_{i=1}^{k} P[a_i \le X_i \le b_i]$$

If the previous holds, the variables are often said to be stochastically independent. If it does not hold, the variables are said to be dependent

$\boxed{\text{Thm 4.4.1}}$ RV $X_1, \quad , X_k$ are independent iff

$$F(X_1, X_2, \quad , X_k) = F_1(X_1)\, F_2(X_2) \quad F_k(X_k)$$

$$f(X_1, X_2, \quad , X_k) = f_1(X_1)\, f_2(X_2) \quad f_k(X_k)$$

where $F_i(X_i)$ and $f_i(X_i)$ are the marginal CDFs & pdfs

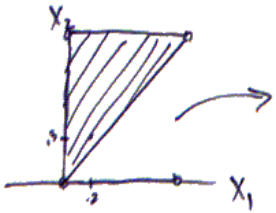
$\boxed{\text{Thm 4.4.2}}$ Two R.V. X_1 & X_2 with joint pdf $f(X_1, X_2)$ are independent Iff

1) The support set $\{(X_1, X_2) \mid f(X_1, X_2) > 0\}$ is a Cartesian product $A \times B$ (The support set is rectangular)

2) The joint pdf can be factored into the product of functions of X_1 & X_2 respectively.

$$f(X_1, X_2) = g(X_1)\, h(X_2)$$

Ex: $f(x_1, x_2) = 8x_1x_2 \quad 0 < x_1 < x_2 < 1$



support set is not rectangular.

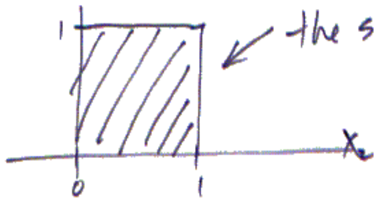
Note: when writing with indicator functions, it is clear that the joint pdf cannot be factored:

$$f(x_1, x_2) = 8x_1x_2 \underbrace{I_{(x_1,1)}(x_2)}_{\text{function of both } x_1 \,\&\, x_2 \text{ and cannot be factored}}$$

$$\neq g(x_1) \cdot h(x_2)$$

Hence, x_1 & x_2 are not indepdent (dependent)

Ex: $f(x_1, x_2) = x_1 + x_2 \quad \begin{matrix} 0 < x_1 < 1 \\ 0 < x_2 < 1. \end{matrix}$



the support set is rectangular.

With indicator functions: $f(x_1, x_2) = (x_1 + x_2) I_{(0,1)}(x_1) I_{(0,1)}(x_2)$

$$= x_1 I_{(0,1)}(x_1) I_{(0,1)}(x_2) + x_2 I_{(0,1)}(x_1) I_{(0,1)}(x_2)$$

$$\neq g(x_1) \cdot h(x_2)$$

Hence, x_1 & x_2 are not independent.

Note Some consequences of Independence

$$E[XY] = E[X]\,E[Y]$$

$$M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$$

Ex: $X \sim GEO(p)$ $\quad X \perp\!\!\!\perp Y$ independent of

$Y \sim GEO(p)$

Then the MGF of $T = X + Y$

$$M_T(t) = M_{X+Y}(t) = M_X(t)\,M_Y(t)$$

$$= \left(\frac{pe^t}{1-qe^t}\right)\left(\frac{pe^t}{1-qe^t}\right) = \left(\frac{pe^t}{1-qe^t}\right)^2$$

$$T \sim NB(2,p)$$

Note: the expectation of a product is the product of the expectations (if X & Y are independent!)

"Is the integral of a product the product of the integrals?"

$$\int f(x) \cdot g(x)\,dx = \int f(x)\,dx \cdot \int g(x)\,dx \,? \quad \text{No!}$$

The rule that applies is the product rule for Integrals.

$$\int u\,dv = uv - \int v\,du$$

$$E[XY] = \int_{\infty}^{\infty}\int_{\infty}^{\infty} xy\, f(x,y)\,dx\,dy$$

$$= \left[\int_{-\infty}^{\infty} x f_1(x)\,dx\right]\left[\int_{-\infty}^{\infty} y f_2(y)\,dy\right]$$

If X & Y are independent, then $f(x,y) = f_1(x) f_2(y)$

$$= \int_{\infty}^{\infty}\int_{\infty}^{\infty} xy\, f_1(x)\, f_2(y)\,dx\,dy = \int_{-\infty}^{\infty} y\, f_2(y)\left[\int_{\infty}^{\infty} x f_1(x)\,dx\right] dy$$

$$\left[\int_{-\infty}^{\infty} x f_1(x)\,dx\right]\left[\int_{-\infty}^{\infty} y f_2(y)\,dy\right] = E(X)\,E(Y)$$

4.5 Conditional Dist.

DEF 4.5.1

Conditional pdf. If X_1 & X_2 are discrete or cont. R.V. with joint pdf $f(x_1, x_2)$, then the cond. pdf of X_2 given $X_1 = x_1$ is

$$f(x_2 | x_1) = \frac{f(x_1, x_2)}{f_1(x_1)}$$

for values such that $f_1(x_1) > 0$, and zero otherwise.

Thm 4.5.1 If X_1 & X_2 are R.V. with joint pdf $f(x_1, x_2)$ and marginals $f_1(x_1)$ $f_2(x_2)$, then

$$f(x_1, x_2) = f_1(x_1) \cdot f(x_2 | x_1)$$
$$= f_2(x_2) \cdot f(x_1 | x_2)$$

If X_1 & X_2 are independent, then

$$f(x_2 | x_1) = f_2(x_2)$$
$$f(x_1 | x_2) = f_1(x_1)$$