DEF Random Var $X_1, X_2, \cdots, X_k$ are said to be independent if for every $a_i < b_i$

$$P[a_1 \le X_1 \le b_1, a_2 \le X_2 \le b_2, \cdots, a_k \le X_k \le b_k]$$

$$= \prod_{i=1}^{k} P[a_i \le X_i \le b_i]$$

If the previous holds, the variables are often said to be stochastically independent. Otherwise they are dependent.

$\boxed{\text{Thm } 4.4.1}$ Random vars $X_1, X_2, \ldots, X_k$ are independent iff

marginal CDF's

$$F(x_1, x_2, \ldots, x_k) = F_1(x_1)F_2(x_2)\cdots F_k(x_k)$$

$$f(x_1, x_2, \cdots, x_k) = f_1(x_1)f_2(x_2)\cdots f_k(x_k)$$

marginal pdfs.

EX:

$$f(x_1,x_2) = \begin{cases} 8x_1x_2, & 0 < \underbrace{x_1 < x_2} < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$= 8x_1x_2 \underbrace{I_{(x_1,1)}(x_2)}_{\text{function that depends on both } x_1 \,\&\, x_2}$$

$\Rightarrow X_1 \,\&\, X_2$ are dependent

EX: $f(x_1, x_2) = x_1 + x_2 \quad , 0 < x_1 < 1 \quad 0 < x_2 < 1$

$$= (x_1 + x_2)\, I_{(0,1)}(x_1)\, I_{(0,1)}(x_2)$$

$$= x_1\, I_{(0,1)}(x_1)\, I_{(0,1)}(x_2) + x_2\, I_{(0,1)}(x_1)\, I_{(0,1)}(x_2)$$

This cannot be factored into the product of marginals. $\Rightarrow$ x_1 & x_2 are not independent.

$\boxed{4.5}$ Conditional Distributions.

DEF: Conditional pdf

$$f(x_2 | x_1) = \frac{f(x_1, x_2)}{f_1(x_1)}$$

for values of x_1 such that $f_1(x_1) > 0$

Thm: If x_1 & x_2 are RV with joint pdf $f(x_1, x_2)$ and marginal pdf's $f_1(x_1)$, $f_2(x_2)$, then

$$f(x_1, x_2) = f_1(x_1)\, f(x_2 | x_1) = f_2(x_2)\, f(x_1 | x_2)$$

If x_1 & x_2 are independent, then

$$f(x_2 | x_1) = f_2(x_2)$$

$$f(x_1 | x_2) = f_1(x_1)$$

Ex: $T = X+Y \sim NB(2,p) \quad f(t) = (t-1)p^2 q^{t-2}$

$X \sim GEO(p) \quad f(x) = pq^{x-1}$

The conditional pdf of T given $X=x$ is

$$f(t|x) = \frac{f(t,x)}{f_1(x)} = \frac{p^2 q^{t-2}}{pq^{x-1}}, \quad t = x+1, x+2, \dots$$

$$= pq^{t-x-1}, \quad t = x+1, x+2, \dots$$

Define $u = t-x$

$$f_{u|x}(u|x) = P[T-X=u \mid X=x] = P[T=u+x \mid X=x]$$

$$= f(u+x|x) = pq^{u+x-x-1} = pq^{u-1}, \quad u=1,2,\dots$$

$$U|X=x \sim GEO(p)$$

The other way around:

$$f(x|t) = \frac{f(x,t)}{f_2(t)} = \frac{\cancel{p^2 q^{t-2}}}{(t-1)\cancel{p^2 q^{t-2}}} = \frac{1}{t-1}$$

$$x = 1, \dots, t-1$$

$$X|t \sim DU(t-1)$$

$\boxed{4.6}$ Random Samples.

DEF: $X_1, X_2, \cdots, X_n$ is a random sample of size n from a pop. with density $f(x)$ if the joint pdf has the form

$$f(x_1, x_2, \dots, x_n) = f(x_1)f(x_2)\cdots f(x_n)$$

(In other words, the random vars are independent and follow a common dist.

X_i iid $N(\mu, \sigma)$

↳ independent identically dist.

Ex: The Lifetime of a certain type of light bulb is assumed to follow the exponential density

$$(*) \quad f(x) = e^{-x} I_{(0,\infty)}(x)$$

A RS of size 2 is obtained. Then

Random Sample

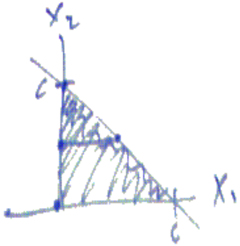
$$f(x_1, x_2) = e^{-(x_1+x_2)} \prod_{i=1}^{2} I_{(0,\infty)}(x_i)$$

Suppose the total lifetime of the two bulbs turned out to be ½ yr

One may wonder whether the result is reasonable given the model

If the model isn't appropriate, another should be chosen. We can find how appropriate by computing probabilities.

$$P[X_1 + X_2 \le c] = \int_0^c \int_0^{c-x_1} e^{-(x_1+x_2)} \, dx_1 \, dx_2$$



$$= 1 - ce^{-c} - e^{-c}$$

For $c = .5$, $P(X_1 + X_2 \leq .5) = .09$

It is unlikely to find the total life time of two bulbs to be .5 years or less, if the true pop model is given by (*)