4.1 Intro:

In many applications, there are more than one random variable of interest.

$$(X_1, X_2, \cdots, X_k)$$

It is convenient to regard these variables as components of a k-dimensional vector $\xrightarrow{\text{capital letters}}$ $X = (X_1, X_2, \cdots, X_k)$, which assumes values $x = (x_1, x_2, \cdots, x_k)$ in a k-dimensional Euclidean space.

small letters

The variable may result from repeated measures or k different characteristics.

4.2 JOINT Discrete Dist

DEF: the joint pdf of the k-dim discrete R.V $X = (X_1, X_2, \cdots, X_k)$ is defined to be

$$f(x_1, x_2, \cdots, x_k) = P\left[\underbrace{X_1 = x_1, X_2 = x_2, \cdots, X_k = x_k}_{\text{Note this is an intersection}}\right]$$

for all possible values of X.

EX: Extended Hypergeometric Dist

Suppose a collection consists of a finite number of items, N, and there are $k+1$ different types of items

M_1 are of type 1

M_2 are of type 2

$\vdots$

M_k are of type k

M_{k+1} are of type $k+1$

Then the pdf of X is

$$f(x_1, x_2, \cdots, x_k) = \frac{\binom{M_1}{x_1}\binom{M_2}{x_2}\cdots\binom{M_k}{x_k}\binom{M_{k+1}}{x_{k+1}}}{\binom{N}{n}}, \quad 0 \le x_i \le M_i$$

and $\sum_{i=1}^{k+1} x_k = n$ and $\sum_{i=1}^{k+1} M_i = N$

Hence $x_{k+1} = n - \sum_{i=1}^{k} x_i$

and $M_{k+1} = N - \sum_{i=1}^{k} M_i$

$X \sim HYP(n, M_1, \cdots, M_k, N)$

Note: if $k=1$, then

$$\frac{\binom{M_1}{x_1}\binom{M_2}{x_2}}{\binom{N}{n}} = \frac{\binom{M_1}{x_1}\binom{N-M_1}{n-x_1}}{\binom{N}{n}} \sim HYP(N, M_1, n$$

$M_2 = N - M_1$

$x_2 = n - x_1$

When selection is with replacemen, then we have the multinomial, a generalization of the binomial.

$X \sim MULT(n, p_1, p_2, \cdots, p_k)$

number of distinguishable permutations.

$$f(x_1, x_2, \dots, x_k) = \frac{n!}{x_1!\, x_2! \cdots x_k!\, x_{k+1}!} p_1^{x_1} p_2^{x_2} \cdots, p_{k+1}^{x_{k+1}}$$

where $0 \le x_i \le n$

$$x_{k+1} = n - \sum_{i=1}^{k} x_i$$

$$p_{k+1} = 1 - \sum_{i=1}^{k} p_i$$

Ex: 4 sided die

Roll 20 times

Record the number of occurrences on each side

Prob of obtaining 4 ones, 10 twos, 3 threes, and 3 fours?

$$P[X_1 = 4, X_2 = 10, X_3 = 3, X_4 = 3] = f(4, 10, 3)$$

$$= \frac{20!}{4!\,10!\,3!\,3!} \left(\frac{1}{4}\right)^4 \left(\frac{1}{4}\right)^{10} \left(\frac{1}{4}\right)^3 \left(1 - \frac{3}{4}\right)^3$$

$$= \frac{775975200}{4^{20}} = \frac{24249225}{34359738368}$$

$$= \boxed{.0007057}$$

$\boxed{\text{Thm 4.2.1}}$

$f(x_1, x_2, \ldots, x_k)$ is a joint pdf iff

$$f(x_1, x_2, \cdots, x_k) \geq 0 \text{ and}$$

$$\sum_{\text{all } x} f(x_1, x_2, \cdot, x_k) = 1$$

Ex. Two dimensional Hypergeometric

Suppose a bin contains 1000 flower seeds

400 are red

200 are pink

400 are white

uppose that 4 seeds are selected at random.

Note: although there are 3 kinds of seeds, only 2 are independent. the last is functionaley dependent on the other two.

White Seeds (x_2)

$f(x_1, x_2)$	0	1	2	3	4
0					
1					
2					
3					
4					

Red Seeds (x_1)

$$f(x_1,x_2) = \frac{\binom{400}{x_1}\binom{400}{x_2}\binom{200}{4-x_1-x_2}}{\binom{1000}{4}}$$

Note: although there are 3 kinds of seeds, only 2 are independent. the last is functionaley dependent on the other two.

what is $P(1 \text{ white}, 1 \text{ red}, 2 \text{ pink})$?

$$= P(1,1) = \frac{\binom{400}{1}\binom{400}{1}\binom{200}{2}}{\binom{1000}{4}}$$

$$= .0769$$

$$f(2,4) = \frac{\binom{400}{2}\binom{400}{4}\binom{200}{-2}}{\binom{1000}{4}} = 0$$

	White Seeds (x_2)				
$f(x_1,x_2)$	0	1	2	3	4
Red Seeds (x_1) 0	.0016	.0127	.0383	.0511	.0254
1	.0127	.0769	.1541	.1022	0
2	.0383	.1541	.1539	0	0
3	.0511	.1022	0	0	0
4	.0254	0	0	0	0