

33

## Special Continuous Dist

We will study several dist.

Uniform

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Gamma, Beta

Exponential

Weibull

Pareto

Normal

Suppose that a cont. R.V. can assume values only in a bounded interval, say  $a < x < b$ , and suppose that  $f(x)$  is constant over that interval

Then  $X \sim \text{UNIF}(a, b)$ 

$$f(x, a, b) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$$

$$= \frac{1}{b-a} I_{(a,b)}(x)$$

The CDF is

$$F(x; a, b) = \begin{cases} 0, & x \leq a \\ \frac{x-a}{b-a}, & a < x < b \\ 1, & x \geq b \end{cases}$$

$$= \frac{x-a}{b-a} I_{(a,b)}(x) + I_{[b,\infty)}(x)$$

The expected value of  $X$  is

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_a^b x \frac{1}{b-a} dx$$

$$= \frac{1}{b-a} \left[ \frac{x^2}{2} \right]_a^b = \frac{1}{2} \frac{b^2 - a^2}{b-a}$$

$$= \frac{1}{2} \frac{(a+b)(b-a)}{b-a} = \frac{b+a}{2}$$

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#### Uniform

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## Gamma Distribution

### Def 3.31 Gamma Function

The gamma function, denoted by  $\Gamma(k)$  for all  $k > 0$ , is given by

$$\Gamma(k) = \int_0^{\infty} t^{k-1} e^{-t} dt$$

The gamma function satisfies the following

$$\Gamma(k) = (k-1)\Gamma(k-1), \quad k > 1$$

$$\Gamma(n) = (n-1)!, \quad n = 1, 2,$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

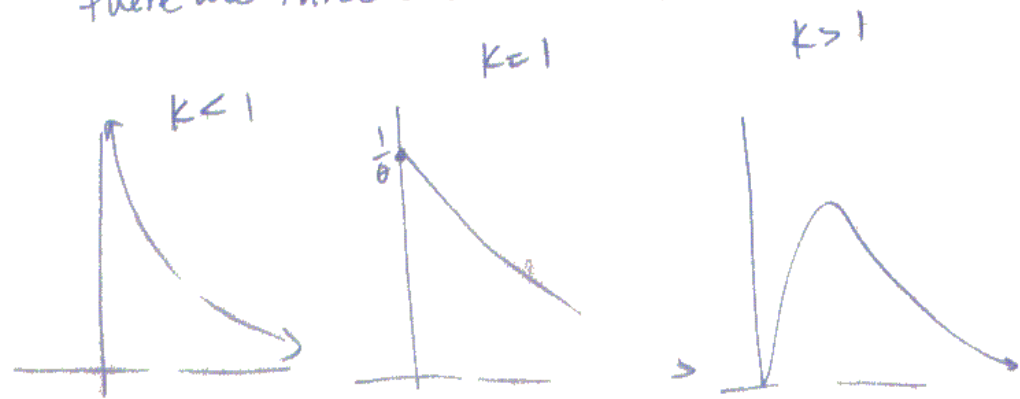
Hint: To prove  $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ , write  $[\Gamma\left(\frac{1}{2}\right)]^2 = \pi$  as a double integral (after substituting  $t = u^2$ ) and change to polar coordinates.

A contin. R.V.  $X$  is said to have the gamma dist with param  $k > 0$ , and  $\theta > 0$  if

$$f(x; \theta, k) = \frac{1}{\theta^k \Gamma(k)} x^{k-1} e^{-\frac{x}{\theta}} I_{(0, \infty)}(x)$$

Note:  $X \sim \text{GAM}(\theta, k)$

The parameter  $k$  is also called a shape parameter because it determines the basic shape of the dist. Specifically, there are three different shapes



The CDF of the gamma is

$$F(x, \theta, k) = \left[ \int_0^x \frac{1}{\Gamma(k)} t^{k-1} e^{-t/\theta} dt \right] I_{(\theta, \infty)}(x)$$

The  $\theta$  parameter is called the Scale parameter. This is important because you don't want your results to depend on scale of measurement used.

A Scale parameter satisfies the relation

$$F(x; \theta) = F\left(\frac{x}{\theta}\right)$$

The CDF  $F(x; \theta, k)$  cannot be generally solved for. However, when  $k$  is an integer, then it can. (This is equiv to doing tab integrals)

**Thm 3.3.2**

If  $X \sim \text{GAM}(\theta, n)$ , where  $n$  is an integer and if  $Y \sim \text{POI}\left(\frac{x}{\theta}\right)$ , then

$$P[X < x] = P[Y \geq n] = 1 - \sum_{i=0}^{n-1} \frac{(x/\theta)^i}{i!} e^{-x/\theta}$$

### The mean of the Gamma Dist

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx = \int_0^{\infty} x \frac{1}{\theta^k \Gamma(k)} x^{k-1} e^{-x/\theta} dx$$

$$u = \frac{x}{\theta} \Rightarrow x = u\theta$$
$$du = \frac{1}{\theta} dx \Rightarrow dx = \theta du$$

$$= \int_0^{\infty} u\theta \frac{1}{\theta^k \Gamma(k)} u^{k-1} \theta e^{-u} du$$

$$= \frac{\theta}{\Gamma(k)} \int_0^{\infty} u^k e^{-u} du = \frac{\theta}{\Gamma(k)} \int_0^{\infty} u^{(k+1)-1} e^{-u} du$$

$$= \frac{\theta \Gamma(k+1)}{\Gamma(k)} = \frac{\theta k \Gamma(k)}{\Gamma(k)} = k\theta$$

### General Moments:

$$E(X^r) = \int_0^{\infty} x^r \frac{1}{\theta^k \Gamma(k)} x^{k-1} e^{-x/\theta} dx$$

$$= \frac{1}{\theta^k \Gamma(k)} \int_0^{\infty} x^{r+k-1} e^{-x/\theta} dx$$

Kernel of Gamma( $\theta, r+k$ )

$$= \frac{1}{\theta^k \Gamma(k)} \frac{\theta^{k+r} \Gamma(k+r)}{1} \underbrace{\int_0^{\infty} \frac{1}{\theta^{k+r} \Gamma(k+r)} x^{k+r-1} e^{-x/\theta} dx}_{\text{integrates to 1 (pdf)}}$$

$$= \frac{\theta^{k+r} \Gamma(k+r)}{\theta^k \Gamma(k)}$$

$$= \frac{\theta^r \Gamma(k+r)}{\Gamma(k)}$$

### 3.3 Cont.

The MGF of Gamma dist is

$$M_X(t) = \left( \frac{1}{1-\theta t} \right)^k$$

#### Special cases

If  $\theta = 2$ ,  $k = \nu/2$ , where  $\nu = \text{degrees of freedom}$ ,

then  $X \sim \chi^2(\nu)$  (chi-squared dist)

#### Exponential Distribution

If  $k=1$ , then GAMMA dist is the EXPONENTIAL dist

If  $k=1$ , then  $X \sim \text{EXP}(\theta)$

$$f(x; \theta) = \frac{1}{\theta} e^{-x/\theta} I_{(0, \infty)}(x)$$

$$F(x) = (1 - e^{-x/\theta}) I_{(0, \infty)}(x)$$

Note that  $\theta$  is a scale parameter.

The exp dist. is useful as a prob. model for lifetimes.

#### The No-Memory property

Thm 3.3.3. For a cont. R.V.  $X$ ,  $X \sim \text{EXP}(\theta)$

iff  $P[X > a+t | X > a] = P[X > t]$  for all  $a > 0$  and  $t > 0$ .

Think of it as this: An old component that still works is just as reliable as a new component.

## Weibull Dist

(A lot like Gamma Dist)

Used for: fatigue & breaking strength of materials  
failure times  
Engineers love it!  
CDF is explicit!

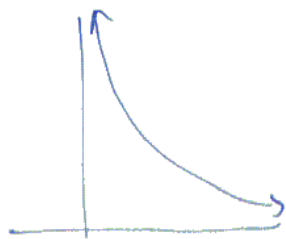
$$X \sim \text{WEI}(\theta, \beta)$$

$$f(x, \beta, \theta) = \frac{\beta}{\theta^\beta} x^{\beta-1} e^{-\left(\frac{x}{\theta}\right)^\beta} I_{(0, \infty)}(x)$$

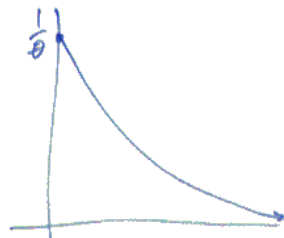
$\beta$  = shape parameter  
 $\theta$  = scale parameter

Shapes:

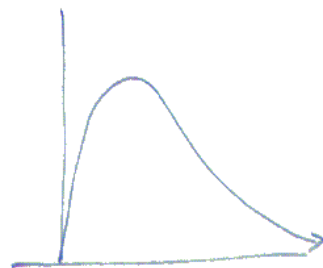
$\beta < 1$



$\beta = 1$



$\beta > 1$



Note the similarities to the Gamma Dist!

$$\text{CDF is } F(x, \theta, \beta) = \left(1 - e^{-\left(\frac{x}{\theta}\right)^\beta}\right) I_{(0, \infty)}(x)$$

Special cases:

If  $\beta = 1$ , then  $X \sim \text{EXP}(\theta)$

If  $\beta = 2$ , then  $X \sim \text{WEI}(\theta, 2)$  and we call it the Rayleigh dist

The mean of the Weibull is

$$E(X) = \theta \Gamma\left(1 + \frac{1}{\beta}\right) \text{ valid if } 1 + \frac{1}{\beta} > 0$$

$\Rightarrow$  always satisfied since  $\beta > 0$

The mgf does not exist in a form that is useful.

The variance is

$$V(X) = \theta^2 \left[ \Gamma\left(1 + \frac{2}{\beta}\right) - \Gamma^2\left(1 + \frac{1}{\beta}\right) \right]$$

The  $100 \times p^{\text{th}}$  percentile is

$$x_p = \theta \left[ -\ln(1-p) \right]^{1/\beta}$$

## Pareto

$$X \sim \text{PAR}(\theta, k)$$

$k = \text{shape param.}$

$\theta = \text{scale param.}$

$$f(x; \theta, k) = \left(\frac{k}{\theta}\right) \left(1 + \frac{x}{\theta}\right)^{-(k+1)} I_{(0, \infty)}(x)$$

$k > 0.$

$$\text{CDF: } F(x; \theta, k) = \left[ 1 - \left(1 + \frac{x}{\theta}\right)^{-k} \right] I_{(0, \infty)}(x)$$

Useful for modeling length of a wire between flaws: Ex: 2.3.2 is an example.

Note the pdf  $f(y) = \left(\frac{k}{a}\right) \left(\frac{y}{a}\right)^{-(k+1)} I_{(a, \infty)}(y)$  is also called a Pareto dist.

The mean & variance are

$$E(X) = \frac{\theta}{k-1}, k > 1 \quad V(X) = \frac{\theta^2 k}{(k-2)(k-1)^2}, k > 2$$

$n^{\text{th}}$  percentile  $x_p = \theta \left[ (1-p)^{1/k} - 1 \right]$



## Normal dist

First published in 1733 as an approx for the sum of Binomial random vars  
It is the single most important dist in statist prob.

$$f(x, \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right\}, \sigma > 0$$

Simply it is denoted by  $X \sim N(\mu, \sigma^2)$

It is also called the Gaussian dist

Verify it integrates to 1:

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = 2 \int_0^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz = \frac{1}{\sqrt{\pi}} \left[ 2 \int_0^{\infty} e^{-u^2} du \right] = \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1$$

$z = \frac{x-\mu}{\sigma}$   
 $dz = \frac{1}{\sigma} dx$   
 $u = \frac{z}{\sqrt{2}}$   
 $du = \frac{1}{\sqrt{2}} dz$

$\phi = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$  is the standard normal pdf

$$\Phi(z) = \int_{-\infty}^z \phi(t) dt \quad \left. \vphantom{\int_{-\infty}^z} \right\} \text{Standard normal CDF}$$

Note:  $\phi(z) = \phi(-z)$  (Even)  
unique max at  $z=0$  & inflection pts at  $\pm 1$

$$\phi'(z) = -z\phi(z)$$

$$\phi''(z) = (z^2 - 1)\phi(z)$$

$$E(z) = 0$$

$$E(z^2) = 1 = \sigma^2$$