

Normal Dist cont.

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

$$\phi'(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \frac{d(-z^2/2)}{dz} = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} (-z) = -z\phi(z)$$

$$\begin{aligned}\phi''(z) &= \frac{d(-z\phi(z))}{dz} = -z\phi'(z) + \phi(z)(-1) \\ &= -z(-z\phi(z)) - \phi(z) \\ &= (z^2 - 1)\phi(z)\end{aligned}$$

$$\phi'''(z) = (z^2 - 1)\phi'(z) + 2z\phi(z) \quad \rightarrow \quad z^2\phi(z) = \phi''(z) + \phi(z)$$

$$= (z^2 - 1)(-z\phi(z)) + 2z\phi(z)$$

$$= (-z^3 + 3z)\phi(z)$$

$$= z(-z^2 + 3)\phi(z)$$

$$\begin{aligned}\text{Max of } \phi(z) \text{ is } \phi'(z) &= 0 \\ -z\phi(z) &= 0 \Rightarrow z = 0\end{aligned}$$

$$\begin{aligned}\text{Inflection pts: } \phi''(z) &= 0 \\ (z^2 - 1)\phi(z) &= 0 \\ z^2 = 1 &\Rightarrow z = \pm 1\end{aligned}$$

$$E(z) = \int_{-\infty}^{\infty} z\phi(z) dz = - \int_{-\infty}^{\infty} \phi'(z) dz$$

$$= - \lim_{b \rightarrow \infty} [\phi(z)]_{-b}^b = \lim_{b \rightarrow \infty} \phi(-b) - \phi(b)$$

$$= 0 - 0 = 0$$

$$E(z^2) = \int_{-\infty}^{\infty} z^2\phi(z) dz = \int_{-\infty}^{\infty} \phi''(z) + \phi(z) dz$$

$$= \int_{-\infty}^{\infty} \phi''(z) dz + \int_{-\infty}^{\infty} \phi(z) dz = \underbrace{\phi'(z)}_{-\infty}^{\infty} + 1 = 1$$

$$\begin{aligned} \text{Thus, } v(z) &= E(z^2) - [E(z)]^2 \\ &= 1 - 0 \\ &= 1 \end{aligned}$$

Thm 3.3.4

If $x \sim N(\mu, \sigma^2)$, then

$$1. \quad z = \frac{x - \mu}{\sigma} \sim N(0, 1)$$

$$2. \quad F_x(x) = \Phi\left(\frac{x - \mu}{\sigma}\right)$$

Standard normal cumulative probabilities are in Table 3 in Appendix C in your book because of symmetry, only positive z values are given. For negative z values, $\Phi(-z) = 1 - \Phi(z)$

Ex: $\Phi(z) = .9772$

Let z_y denote the y^{th} percentile of the standard normal, which means

$$\Phi(z_y) = y$$

For example, $\Phi(z_{.90}) = .90 \Rightarrow z_{.90} = 1.282$

It is often useful to consider normal prob in terms of standard deviations from the mean.

$$\begin{aligned} P[\mu - 2\sigma < x < \mu + 2\sigma] &= F_x(\mu + 2\sigma) - F_x(\mu - 2\sigma) \\ &= \Phi\left(\frac{\mu + 2\sigma - \mu}{\sigma}\right) - \Phi\left(\frac{\mu - 2\sigma - \mu}{\sigma}\right) = \Phi(2) - \Phi(-2) \\ &= .9772 - (.0228) = .9544 \end{aligned}$$

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Standard normal cumulative probabilities are in Table 3 in Appendix C in your book.

Because of symmetry, only positive z values are given. For negative z values, $\Phi(-z) = 1 - \Phi(z)$

Ex: $\Phi(z) = .9772$

Let z_γ denote the γ^{th} percentile of the standard normal, which means

$$\Phi(z_\gamma) = \gamma$$

For example, $\Phi(z_{.90}) = .90 \Rightarrow z_{.90} = 1.282$

It is often useful to consider normal prob in terms of standard deviations from the mean

$$\begin{aligned} P[\mu - 2\sigma < x < \mu + 2\sigma] &= F_x(\mu + 2\sigma) - F_x(\mu - 2\sigma) \\ &= \Phi\left(\frac{\mu + 2\sigma - \mu}{\sigma}\right) - \Phi\left(\frac{\mu - 2\sigma - \mu}{\sigma}\right) = \Phi(2) - \Phi(-2) \\ &= .9772 - (.0228) = .9544 \end{aligned}$$

Note that the normal random variable can still be used as a reasonable model for a random variable that takes on only positive values, if very little prob. is associated with the neg. values. (Another choice is to use a truncated normal model.)

Thm 3.3.5

If $X \sim N(\mu, \sigma^2)$, then

$$(1) M_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2} = \exp\left\{\mu t + \frac{1}{2}\sigma^2 t^2\right\}$$

$$(2) E(X-\mu)^{2r} = \frac{(2r)! \sigma^{2r}}{r! 2^r}, \quad r=1, 2, \dots$$

$$(3) E(X-\mu)^{2r-1} = 0, \quad r=1, 2, \dots$$

Proof:

- (1) Complete the square in the exponent and use the "kernel method".
- (2) Expand the MGF in a Maclaurin series.
- (3)

3.4 Location-Scale Parameters.

$F_0(z)$ represents a completely specified pdf (Does not depend on parameter)

$f_0(z)$ is the pdf corresponding to F_0

DEF

Location Parameters

$\eta \rightarrow$ location parameter for a dist X if

$$F(x, \eta) = F_0(x - \eta)$$

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Ex $f(x, \eta) = e^{-(x-\eta)} \quad x > \eta$

$$x > \eta \Rightarrow \eta < x < \infty \Rightarrow I_{(\eta, \infty)}(x)$$

$$0 < x - \eta < \infty \Rightarrow I_{(0, \infty)}(x - \eta)$$

Thus, $f(x, \eta) = e^{-(x-\eta)} I_{(0, \infty)}(x - \eta)$

$$f_0(x) = e^{-x} I_{(0, \infty)}(x) \Rightarrow f_0(x - \eta) = e^{-(x-\eta)} I_{(0, \infty)}(x - \eta)$$

It is common for the location parameter to be a measure of central tendency such as the median or mean (doesn't always happen)

DEF: Scale parameters

$\theta > 0 \Rightarrow$ scale parameter for a R.V X if

$$F(x; \theta) = F_0\left(\frac{x}{\theta}\right)$$

in other words $f(x, \theta) = f_0\left(\frac{x}{\theta}\right) \left(\frac{1}{\theta}\right) = \frac{1}{\theta} f_0\left(\frac{x}{\theta}\right)$

Ex: $X \sim \text{EXP}(\theta) \Rightarrow \theta$ is a scale parameter
 $X \sim N(\mu, \sigma^2) \Rightarrow \sigma$ is a scale parameter
Often σ , the std. dev. is a scale parameter! (not always)
 $X \sim \text{WEI}(\theta, 2) \Rightarrow \theta$ is a scale parameter, but it is not the std. deviation

DEF 3.4.3

Location Scale Parameters

η and $\theta > 0$ are called location-scale parameters if the CDF has the form

$$F(x, \theta, \eta) = F_0\left(\frac{x - \eta}{\theta}\right)$$

or

$$f(x, \theta, \eta) = \frac{1}{\theta} f_0\left(\frac{x - \eta}{\theta}\right)$$

Ex: Cauchy dist.

$$f_0(z) = \frac{1}{\pi} \frac{1}{1+z^2}$$

$$\begin{aligned} f(x; \theta, \eta) &= \frac{1}{\theta} f\left(\frac{x - \eta}{\theta}\right) \\ &= \frac{1}{\pi\theta} \frac{1}{1 + \left(\frac{x - \eta}{\theta}\right)^2} \end{aligned}$$

$$X \sim \text{CAU}(\theta, \eta)$$

Note: $E(X)$ d.n.e.
 $V(X)$ d.n.e.

So η & θ do not represent the mean & std dev.