$\boxed{2.5}$ Cont

Properties of Moment Generating functions

$\boxed{\text{Thm } 2.5.2}$

If $Y = aX + b$, then $M_Y(t) = e^{bt} M_X(at)$

Proof:

$$M_Y(t) = E[e^{tY}] = E\left[e^{t(aX+b)}\right] =$$

$$= E\left[e^{tb} e^{X(at)}\right] = e^{tb} \underbrace{E\left[e^{X(at)}\right]}_{M_X(at)}$$

$$= e^{tb} M_X(at)$$

It can be shown the MGF's uniquely determine a distribution (if MGF exists)

$\boxed{\text{Application of Thm } 2.5.2}$

$Y = X - \mu$

Then

$$E[(X-\mu)^r] = \frac{d^r}{dt^r}\left[e^{-\mu t} M_X(t)\right]\Big|_{t=0}$$

$\boxed{\text{Thm } 2.5.3}$ Uniqueness

If X_1 & X_2 have respective CDFs $F_1(x)$ and $F_2(X)$ and MGFs $M_1(t)$ and $M_2(t)$ then $F_1(x) = F_2(X)$ for all real x iff

$M_1(t) = M_2(t)$ for all t in some neighborhood of 0
($-h < t < h$ for some $h > 0$)

Factorial Moments

(DEF 2.52)
The r^{th} factorial moment of X is

$$E[X(X-1) \quad (X-r+1)]$$

and the factorial moment generating function (FMGF) of X is

$$G_X(t) = E[t^X]$$

if this expectation exists for all t in a neighborhood of 1 ($1-h < t < 1+h$ and $h>0$)

The factorial moment generating function is also called the prob. generating function because for non-negative integer valued random var. X,

$$P[X=r] = \frac{G_X^{(r)}(0)}{r!}$$

Note the relationship:

$$G_X(t) = E[t^X] = E\left[e^{\ln t^X}\right] = E\left[e^{(\ln t)X}\right] = M_X(\ln t)$$

$\boxed{\text{Thm } 2.5.4}$ If X has FMGF, $G_X(t)$, then

$$G_X'(1) = E(X)$$

$$G_X''(1) = E[X(X-1)]$$

$$\vdots$$

$$G_X^{(r)}(1) = E[X(X-1)\cdots(X-r+1)]$$

Note that regular moments can be derived from the factorial moments. For example:

$$E[X(X-1)] = E(X^2 - X) = E(X^2) - E(X)$$

$$\Rightarrow E(X^2) = E[X(X-1)] + E(X)$$

$$= G_X''(1) + G_X'(1)$$