

24 Cont.

Thm 2.4.8 Let $\mu = E(X)$ and $\sigma^2 = \text{Var}(X)$

If $\sigma^2 = 0$, then $P\{X = \mu\} = 1$

Proof. If $X \neq \mu$ for some observed value x , then $|x - \mu| \geq \frac{1}{i}$ for some integer $i \geq 1$ and

Thus,
$$\{X \neq \mu\} = \bigcup_{i=1}^{\infty} \left\{ |X - \mu| \geq \frac{1}{i} \right\}$$

Using Boole's Inequality

$$P\{X \neq \mu\} \leq \sum_{i=1}^{\infty} P\left\{ |X - \mu| \geq \frac{1}{i} \right\} \leq \sum_{i=1}^{\infty} i^2 \underbrace{\frac{\sigma^2}{i^2}}_{=0} = 0 \Rightarrow P\{X \neq \mu\} \leq 0$$

Chebychev's Inequality. $\Rightarrow P\{X \neq \mu\} = 0$

$$P\{|X - \mu| \geq k\sigma\} \leq \frac{1}{k^2}$$

$$\begin{matrix} k=0 \rightarrow \\ k=1 \rightarrow 1 = \frac{1}{1} = 1 \\ k=2 \rightarrow \frac{1}{4} = 0.25 \\ \dots \end{matrix}$$

$$\rightarrow \text{So } P\{X = \mu\} = 1 - P\{X \neq \mu\} = 1$$

Note that this dist is called a degenerate dist (A dist that concentrates all the prob at one pt)

Approximate Mean and Variance

If a function of a Random Var, $h(X)$ can be expanded in a Taylor series, then an expression for the approx mean and variance can be obtained in terms of the mean & Var of X .

DEFine $\mu = E(x)$.

(*) $H(x) \doteq$ ^{approx equal to} $H(\mu) + H'(\mu)(x-\mu) + \frac{1}{2} H''(\mu)(x-\mu)^2$

$$E[H(x)] \doteq \underbrace{E[H(\mu)]}_{H(\mu)} + H'(\mu) \underbrace{\left[E(x - \underbrace{\mu}_{\mu}) \right]}_{\mu - \mu = 0} + \frac{1}{2} H''(\mu) \underbrace{E(x-\mu)^2}_{\text{Var}(x) = \sigma^2}$$

$$E[H(x)] \doteq H(\mu) + \frac{1}{2} H''(\mu) \sigma^2$$

$$\text{Var}(ax+b) = a^2 \text{Var}(x)$$

$$\text{Var}(H(x)) = \text{Var} \left(\underbrace{H(\mu)}_b + \underbrace{H'(\mu)}_a (x-\mu) \right) = [H'(\mu)]^2 \sigma^2$$

The accuracy of these approx depends primarily on the nature of $H(x)$ as well as the amount of variability in x .

[25] Moment Generating functions

DEF 2.5.1 If X is a R.V., then the expected value

$$M_X(t) = E[e^{tx}]$$

is called the moment generating function (mgf) of X if the expected value exists for all values of t in some interval of the form $-h < t < h$ for some $h > 0$.

Sometimes, the x is dropped and we write

$M(t)$ instead of

$M_X(t)$

Ex: Assume X is discrete & finite valued.

The mgf is

$$M_X(t) = E[e^{tX}] = \sum_{i=1}^m e^{tx_i} f(x_i)$$

The first derivative of $M_X(t)$ wrt t is

$$M'_X(t) = \sum_{i=1}^m x_i e^{tx_i} f(x_i)$$

The r^{th} derivative of $M_X(t)$ wrt t is

$$M_X^{(r)}(t) = \sum_{i=1}^m (x_i)^r e^{tx_i} f(x_i)$$

Note:

$$M_X^{(r)}(0) = \sum_{i=1}^m x_i^r f(x_i) = E[X^r] = \mathcal{M}'_r$$

Thm 2.5.1 If the mgf exists, then

$$\mathcal{M}'_r = E(X^r) = M_X^{(r)}(0) \text{ for } r=1,2,\dots$$

and

$$M_X(t) = \sum_{r=0}^{\infty} \frac{E(X^r)}{r!} t^r$$

Ex: Consider the cont. P.V. X with pf

$$f(x) = \frac{1}{4} x e^{-x/2}, \quad x > 0$$

The mgf is

$$M_X(t) = E[e^{tX}] = \int_0^{\infty} e^{tx} \frac{1}{4} x e^{-x/2} dx$$

$$= \frac{1}{4} \int_0^{\infty} x e^{(t-\frac{1}{2})x} dx = \frac{1}{4} \int_0^{\infty} e^u \frac{du}{t-\frac{1}{2}} \left(\frac{u}{t-\frac{1}{2}} \right)$$

$u = (t-\frac{1}{2})x = tx - \frac{1}{2}x$
 $du = (t-\frac{1}{2})dx$
 $dx = \frac{du}{t-\frac{1}{2}}$

$t < \frac{1}{2}$

$$= \frac{1}{4\left(t-\frac{1}{2}\right)^2} \int_0^{-\infty} u e^u du$$

$u=u \quad dv=e^u du$
 $du=du \quad v=e^u$

$$= \frac{1}{4\left(t-\frac{1}{2}\right)^2} \left[u e^u - \int e^u du \right]_0^{-\infty}$$

$$= \frac{1}{4\left(t-\frac{1}{2}\right)^2} \left[u e^u - e^u \right]_0^{-\infty}$$

$$= \frac{1}{4\left(\frac{2t-1}{2}\right)^2} = \frac{1}{(2t-1)^2} = \left(\frac{1}{1-2t}\right)^2$$

, $t < 1/2$

Ex.

A discrete P.V. with pdf

$$f(x) = .7^x, \quad x=1, 2, \dots$$

$$M_X(t) = E[e^{tx}] = \sum_{x=1}^{\infty} e^{tx} (.7)^x$$

$$= \sum_{x=1}^{\infty} (.7e^t)^x = \frac{.7e^t}{1 - .7e^t}$$

$$|.7e^t| < 1 \Rightarrow .7e^t < 1$$

$$e^t < \frac{10}{7}$$

$$t < \ln\left(\frac{10}{7}\right)$$