$\boxed{2.3}$ Cont

Find the median of $F(x) = 1 - e^{-\left(\frac{x}{3}\right)^2}$, $x > 0$.

Median is where $F(x_{.5}) = .5$

$$1 - e^{-\left(\frac{x_{.5}}{3}\right)^2} = .5$$

$$1 - .5 = e^{-\left(\frac{x_{.5}}{3}\right)^2}$$

$$.5 = e^{-\left(\frac{x_{.5}}{3}\right)^2}$$

$$\ln(.5) = -\left(\frac{x_{.5}}{3}\right)^2$$

$$3\sqrt{-\ln(.5)} = x_{.5} \quad (\text{since } x_{.5} > 0)$$

$$x_{.5} = 3\sqrt{\ln 2} \approx 974 \text{ months}$$

$\boxed{\text{Def:}}$ If the pdf has a unique max at $x = m_0$, (e.g. $\max_{\text{all } x} f(x) = f(m_0)$), then $m_0$ is called the mode of $x$

Ex: The pdf of the previous example is

$$f(x) = \left(\frac{2}{9}\right) x e^{-\left(\frac{x}{3}\right)^2}, \quad x > 0$$

$$f'(x) = 0 = \frac{2}{9}\left[x e^{-\left(\frac{x}{3}\right)^2}\left(-\frac{2}{9}x\right) + e^{-\left(\frac{x}{3}\right)^2}\right]$$

$$= \frac{2}{9} e^{-\left(\frac{x}{3}\right)^2}\left[-\frac{2}{9}x^2 + 1\right] = 0$$

$$-\frac{2}{9}x^2 + 1 = 0 \Rightarrow x^2 = \frac{9}{2}$$

$$\boxed{x = \frac{3}{\sqrt{2}} = 2.121 \text{ months}}$$

In general, the mean, median, and mode are all different, but there are cases when they agree.

## 2.4 Some properties of Expected Values

Random Variable $X$. Consider $Y = u(X)$
(A function of a Random Variable is another random variable.

**Thm 2.4.1** If $X$ is a Random variable with pdf $f(x)$ and $u(x)$ is a real valued function whose domain includes all the possible values of $X$, then

$$E[u(x)] = \begin{cases} \int_{-\infty}^{\infty} u(x) f(x)\,dx & \text{, if } X \text{ is continuous} \\ \sum_{\text{all } x} u(x) f(x) & \text{, if } X \text{ is discrete.} \end{cases}$$

**Thm 2.4.2** Linearity of the Expectation Operator.

If $X$ is a R.V. with pdf $f(x)$, $a, b$ are constants, and $g(x)$ & $h(x)$ are real-valued functions whose domains include all the possible values of $X$, then

$$E[a g(x) + b h(x)] = a E g(x) + b E h(x)$$
$$= a E[g(x)] + b E[h(x)]$$

**DEF 2.4.1** The variance of a R.V. $X$ is given by

$$var(X) = E(X-\mu)^2 = E[(X-\mu)^2]$$

Note: $\sigma^2$, $\sigma_X^2$, and $V(X)$ are all common notations for $var(X)$. Further, $\sigma = \sqrt{var(X)}$
* $\sigma_X$ is called the standard deviation.

$\boxed{23}$ cont

DEF A dist with pdf $f(x)$ is said to be symmetric about c if

$$f(c-x) = f(c+x) \text{ for all } x.$$

Asymmetric distributions are called skewed dist.

Mixed Distribution

It is possible to have a random variable whose dist. is neither purely discrete nor continuous

A prob dist. for a RV X is of mixed type if the CDF has the form

$$F(x) = aF_d(x) + (1-a)F_c(x)$$

$F_d$ is the CDF of a discrete R.V and $F_c$ is the CDF of a continuous R.V. and $0<a<1$

Corollary

$Var(X) = E(X^2) - \mu^2$, where $\mu = E(X)$.

Proof: $Var(X) = E(X-\mu)^2$ (by def)

$= E[X^2 - 2X\mu + \mu^2]$

$= E(X^2) - 2\mu \underbrace{E(X)}_{\mu} + \mu^2$

$= E(X^2) - 2\mu^2 + \mu^2$

$= E(X^2) - \mu^2$

$= E(X^2) - [E(X)]^2$

Note: The variance (or std. dev) provides a measure of the "spread."

DEF: The $k^{th}$ moment about the orgin of a R.V. X is

$$\mu'_k = E(X^k)$$

and the $k^{th}$ moment about the mean is

$$\mu_k = E[(X-\mu)^k] = E(X-\mu)^k$$

Note: The mean $\mu$ is the first moment about the orgin $\mu = \mu'_1$

The variance $\sigma^2$ is the second moment about the mean $\sigma^2 = E(X-\mu)^2 = \mu_2$

$\boxed{\text{Thm 7.4.4}}$ If X is a R.V. and a, b are const. then

$$Var(aX+b) = a^2 Var(X)$$

**Note:** the mean absolute deviation is defined as

$$E|X-\mu| \qquad (MAD)$$

**Thm 2.4.5** If a dist is symmetric about mean $\mu$, then the third moment about the mean is 0. (e.g. $\mu_3 = 0$)

$\boxed{\text{Thm. 2.4.6}}$ If $X$ is a R.V. $u(X)$ is a non-neg. real valued function, then for any positive constant $c>0$,

$$P[u(X) \geq c] \leq \frac{E[u(X)]}{c}$$

**Markov inequality** set $u(x) = |X|^r$, $r>0$

then thm 2.4.6 gives

$$P[|X| \geq c] \leq \frac{E[|X|^r]}{c^r}$$

$\boxed{\text{Thm 2.4.7}}$ Chebychev's Inequality

If $X$ is R.V with mean $\mu$ and variance $\sigma^2$, then for any $k>0$,

$$P[|X-\mu| \geq k\sigma] \leq \frac{1}{k^2}$$

proof: $u(x) = (X-\mu)^2$

An alternative form is

$$P[|X-\mu| < k\sigma] \geq 1 - \frac{1}{k^2}$$