$\boxed{22}$ Cont.

DEF. The Cumulative dist function (CDF) of a random var X is defined for any real number x by

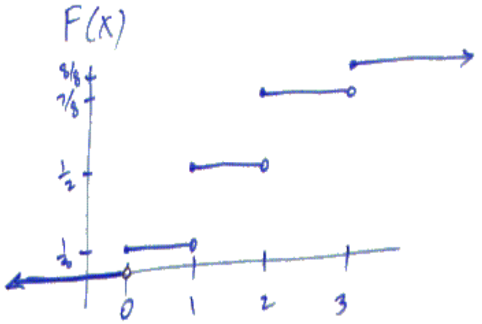
$$F(x) = P[\underset{\text{big}}{X} \le \underset{\text{small}}{x}]$$

For example,

x	0	1	2	3
$f(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$\Rightarrow$

x	0	1	2	3
$F(x)$	$\frac{1}{8}$	$\frac{4}{8}$	$\frac{7}{8}$	$\frac{8}{8}$



often, we refer to the CDF as the distribution function.

Notation:

$$X \sim f(x) \quad \text{or} \quad X \sim F(X)$$

means that X has the pdf $f(x)$ and CDF $F(x)$

Note the $F(X)$ is a non decreasing step function

Thm 2.2.2 Let X_i be a discrete R.V. with pdf $f(x)$ and CDF $F(x)$. If the possible values of X are indexed in increasing order

$$x_1 < x_2 < x_3 < \quad < x_n < \cdots$$

$$f(x_1) = F(x_1)$$

$$f(x_i) = F(x_i) - F(x_{i-1})$$

Furthermore, if $x < x_1$, then $F(x) = 0$ and for any other real x,

$$F(x) = \sum_{x_i \le x} f(x_i)$$

Thm A function $F(x)$ is a CDF for some R.V. iff it satisfies the following

1. $\lim_{x \to -\infty} F(x) = 0$
2. $\lim_{x \to \infty} F(x) = 1$
3. $\lim_{h \to 0^+} F(x+h) = F(x)$ (Right continuous)
4. If $a < b$, then $F(a) \le F(b)$

$\boxed{\text{DEF 2.2.3}}$ If X is a discrete R.V with pdf $f(x)$, then the expected value of X is defined by

$$E(X) = \sum_{x} x f(x)$$

Expected value is also called the expectation or the mean

Ex: Expected value of

$X = \#$ of heads in 3 tosses.

x	0	1	2	3
$f(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$E(X) = \sum_x x f(x) = 0\left(\frac{1}{8}\right) + 1\left(\frac{3}{8}\right) + 2\left(\frac{3}{8}\right) + 3\left(\frac{1}{8}\right)$$

$$= \frac{3}{8} + \frac{6}{8} + \frac{3}{8}$$

$$= \frac{12}{8} = 1.5$$

Ex: $X =$ max of two dice

x	1	2	3	4	5	6
$f(x)$	$\frac{1}{36}$	$\frac{3}{36}$	$\frac{5}{36}$	$\frac{7}{36}$	$\frac{9}{36}$	$\frac{11}{36}$

$$E(X) = 1\left(\frac{1}{36}\right) + 2\left(\frac{3}{36}\right) + 3\left(\frac{5}{36}\right) + 4\left(\frac{7}{36}\right) + 5\left(\frac{9}{36}\right) + 6\left(\frac{11}{36}\right)$$

$$= \frac{161}{36} = 4.4722$$

2.3 Cont R.V

DEF: A random variable X is called a continuous R.V if there is a function $f(x)$, called the pdf of X, such that the CDF can be represented as

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

Note $f(x) = \frac{d}{dx} F(x) = F'(x)$

Further, Note the def. of $F(x)$

$$F(x) = P[X \le x]$$

$$P[a < X \le b] = P[X \le b] - P(X \le a)$$
$$= F(b) - F(a)$$

Also,

$$P[a \le X \le b] = P(X \le b) - P(X < a)$$
$$= P(X \le b) - P(X \le a) + P[X = a]$$
$$= F(b) - F(a) + \underbrace{P[X = a]}_{\text{what is this?}}$$

For a cont. prob dist.,

$P[X=c]=0$, Hence,

$P[a<X<b] = P[a\le X<b] = P[a<X\le b] = P(a\le X\le b)$

Thm: 2.3.1

A function $f(x)$ is a pdf for some continuous R.V. X iff

1) $f(x)\geq 0$ for all $x\in(-\infty,\infty)$

2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Compare to discrete.

1) $f(x_i)\geq 0$ for all x_i

2) $\sum_{\text{all } i} f(x_i) = 1$

Ex: Suppose $f(x)=\begin{cases} c(1+x)^{-3}, & x>0 \\ 0, & x\le 0 \end{cases}$

Where c is a constant.

By prop 1, $c>0$ in order for $f(x)\geq 0$.

By prop 2,

$$1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} c(1+x)^{-3}\,dx$$

$$= \lim_{b\to\infty}\left[\frac{c(1+x)^{-2}}{-2}\right]_0^b$$

$$= c \lim_{b\to\infty} \frac{1}{-2(1+b)^2} + \frac{c}{2}$$

$$= \frac{c}{2} = 1 \Rightarrow \boxed{c=2}$$

Hence

$$f(x) = \begin{cases} 2(1+x)^{-3}, & x > 0 \\ 0, & x \le 0 \end{cases}$$

The cdf for x is

$$F(x) = P[X \le x] = \int_{-\infty}^{x} f(t)\,dt$$

Suppose $x \le 0$, then

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} 0\,dt = 0$$

Suppose $x > 0$, then

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \underbrace{\int_{-\infty}^{0} f(t)\,dt}_{0} + \int_{0}^{x} f(t)\,dt$$

$$= \int_{0}^{x} f(t)\,dt = \int_{0}^{x} 2(1+t)^{-3}\,dt = \left.\frac{\cancel{2}(1+t)^{-2}}{-\cancel{2}}\right]_{0}^{x}$$

$$= 1 - \frac{1}{(1+x)^2}$$

$$F(x) = \begin{cases} 0, & \text{if } x \le 0 \\ 1 - \dfrac{1}{(1+x)^2}, & x > 0 \end{cases}$$

Note: $f(x)$ is <u>not</u> a probability, although it can be used to assign probability to arbitrarily small intervals.

Note: $$P[a \le x \le b] = \int_a^b f(x)\,dx = \underset{\substack{\uparrow \\ \text{CDF}}}{F}(b) - F(a)$$

<u>DEF</u>: If X is cont with pdf $f(x)$, the expected value of X is

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

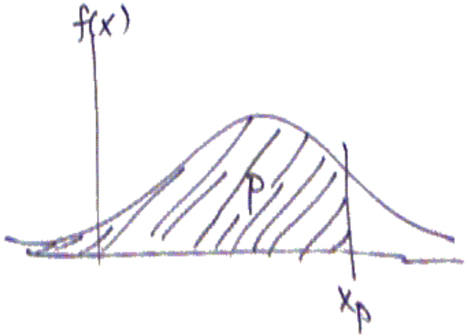
if the integral is absolutely convergent. Otherwise, we say the $E(x)$ does not exist.

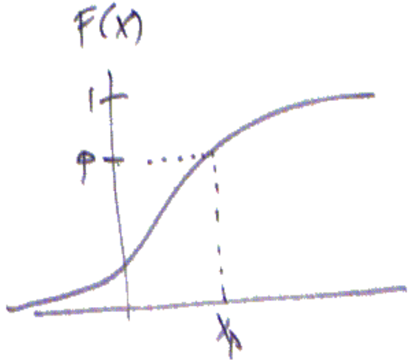
$\boxed{\text{DEF 2.3.3}}$

If $0 < p < 1$, then a $100 \times p^{th}$ <u>percentile</u> of the distribution of a continuous R.V X is a solution x_p to the equation

$$F(x_p) = p$$

Ex:





We can also think of this in terms of quantiles (e.g. the 97th percentile is the .97 quantile)

The median of the dist X is a 50th percentile, denoted by $X_{.5}$ or m.

Ex X is the dist of lifetimes (in months) of a particular component

the CDF is $F(x) = 1 - e^{-\left(\frac{x}{2}\right)^2}$, $x > 0$

Find the median!