

1.3

DEFINITION OF PROB.

$P(A)$ is a set function (that is, it is a function whose domain is a collection of sets, and the range is a subset of the real numbers.

$$P(A): B \rightarrow [0, 1]$$

↑
collection of sets.

- Not all set functions are suitable for assigning prob.

DEF: For a given experiment,

S denotes the sample space

A, A_1, A_2, \dots represent possible events.

A set function $P(A)$ is called the prob of A if the following are satisfied:

$$0 \leq P(A) \text{ for every } A$$

$$P(S) = 1$$

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) \quad \text{if } A_i \cap A_j = \emptyset \text{ whenever } i \neq j$$

First consequence:

$$P(\emptyset) = 0$$

$$S \cup \emptyset = S$$

$$P(S \cup \emptyset) = P(S)$$

$$P(S) + P(\emptyset) = P(S)$$

$$\boxed{P(\emptyset) = 0} \checkmark$$

Probability in Discrete Spaces

The assignment of prob in the case of discrete spaces can be reduced to assigning probability to elementary events.

Let $\{e_i\}$ be an elementary event.

$$P(\{e_i\}) = p_i$$

So, to satisfy the axioms of prob (defn Prob)

$$p_i \geq 0 \text{ for all } i.$$

$$\sum_{\text{all } i} p_i = 1$$

EX: Coin toss

$$S = \{HH, HT, TH, TT\}$$

\Rightarrow prob of each event is $1/4$.

$$P(HH) = 1/4 = P(HT) =$$

$$P(TH) = P(TT)$$

$X = \#$ of heads in 2 toss

$$S = \{0, 1, 2\} \quad \frac{N!}{k!}$$

$$\Rightarrow P(\text{each}) = 1/3$$

$$P(X=1) = P(HT) + P(TH)$$

$$= \frac{1}{4} + \frac{1}{4}$$

$$= 1/2$$

$$P(0) = P(2) = 1/4.$$

Classical Prob

- finite number of outcomes.
- equally likely outcomes

$$S = \{e_1, e_2, \dots, e_N\}$$

$$P_1 = P_2 = \dots = P_N = \frac{1}{N}$$

$$P(A) = \frac{\overset{\text{any event}}{n(A)}}{N} = \frac{\text{\# of ways } A \text{ can occur}}{\text{total size of sample space.}}$$

Random Selection

If an object is chosen from a finite collection of distinct objects in such a manner that each object has same prob of being chosen, then we say that the object was chosen at random.

1.4 Some properties of Prob.

Thm 1.4.1

If A is an event and A' is its complement, then

$$P(A) = 1 - P(A')$$

Proof: $A \cup A' = S$

$$P(A \cup A') = P(S) = 1$$

$$P(A) + P(A') = 1$$

$$P(A) = 1 - P(A')$$

Ex: $P(\text{At least one girl in 3 births})$

$A = \text{At least one girl in 3 births}$

$A' = \text{no girls in 3 births}$

$X = \# \text{ of girls in 3 births}$

$$P(A) = P(X=1) + P(X=2) + P(X=3)$$

$$= 1 - P(A')$$

$$= 1 - P(X=0)$$

$$= 1 - \frac{1}{8} = \frac{7}{8}$$

Thm 1.4.2 For any event A , $P(A) \leq 1$

proof: Not, since $P(A') \geq 0$ (Axiom)

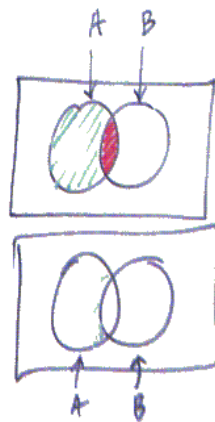
$$1 - P(A) \geq 0 \Rightarrow 1 \geq P(A)$$

Thm 1.4.3 For any two events A, B

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Note: $A = \underline{(A \cap B)} \cup \underline{(A \cap B')}$ —

$$A \cup B = \underline{(A \cap B)} \cup B$$



$$P(A) = P(A \cap B) + P(A \cap B') \quad (\text{Prob of the union of disjoint events})$$

$$P(A \cup B) = P(A \cap B') + P(B)$$

$$= P(A) - P(A \cap B) + P(B)$$

$$= P(A) + P(B) - P(A \cap B)$$

Thm 1.4.4 for any three events

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

Proof:

$$P(A \cup D) = P(A) + P(D) - P(A \cap D)$$

$$= P(A) + P(B \cup C) - P(A \cap (B \cup C))$$

$$= P(A) + P(B) + P(C) - P(B \cap C) - P((A \cap B) \cup (A \cap C))$$

$$[P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)]$$

Thm 1.4.5

If $A \subset B$, then $P(A) \leq P(B)$

Thm 1.4.6 Boole's Inequality.

A_1, A_2, \dots , is a sequence of events, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} P(A_i)$$

Thm 1.4.7 Bonferroni's Inequality

$$P\left(\bigcap_{i=1}^k A_i\right) \geq 1 - \sum_{i=1}^k P(A_i^c)$$

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$A = \text{At least one girl in 3 births}$

$A' = \text{no girls in 3 births}$

$X = \# \text{ of girls in 3 births}$

$$P(A) = P(X=1) + P(X=2) + P(X=3)$$

$$= 1 - P(A')$$

$$= 1 - P(X=0)$$

$$= 1 - \frac{1}{8} = \frac{7}{8}$$

Thm 1.4.2 For any event A , $P(A) \leq 1$

proof: Not, since $P(A') \geq 0$ (Axiom)

$$1 - P(A) \geq 0 \Rightarrow 1 \geq P(A)$$

Thm 1.4.3 For any two events A, B

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Note: $A = \underline{(A \cap B)} \cup \underline{(A \cap B')}$ —

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