9.2 The Euler Methods

Euler's Method

Use lines to approximate y.

The Linearization of a function is

$$L(x) = y(x_0) + y'(x_0)(x - x_0)$$

But,

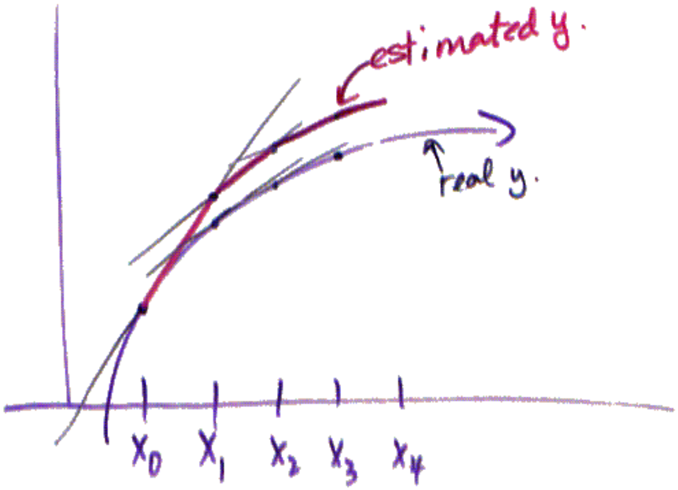
$$\frac{dy}{dx} = f(x,y) = y'(x)$$

Make the spacing equal.

$$h = x - x_0$$

$$\underbrace{L(x)}_{y_1} = \underbrace{y(x_0)}_{y_0} + \underbrace{y'(x_0)}_{f(x_0,y_0)}\underbrace{(x-x_0)}_{h}$$

$$\boxed{y_{n+1} = y_n + f(x_n, y_n)\,h}$$



Use Euler's Method to Solve

$$y' = \underbrace{-y + e^{2x}}_{f(x,y)} \qquad y(0)=1$$

To test these methods, we want to get the exact answer, so we can compare it.

$$\frac{dy}{dx} + y = e^{2x}$$

⋮ and then a miracle occurs

⋮

$$\boxed{y = \frac{1}{3}e^{2x} + \frac{2}{3}e^{-x}}$$

$h = .1$

k	x_k	y_k	y_{exact}	Abs Error	Rel Error
0	0	1	1	0	0
1	.1	1	1.01036	.01036	.01036
2	.2	1.02214	1.04310	.02096	.02009
3	.3	1.06911	1.10125	.03214	.02919
4	.4	1.14441	1.18873	.04432	.03728
5	.5	1.25252	1.31045	.05792	.04420

Abs Error = |true value − approx|

$$\text{Relative Error} = \frac{\text{Abs error}}{|\text{true value}|}$$

Rel error is 4.4%.

$$y_{n+1} = y_n + f(x_n, y_n)\,h$$

$$y_{n+1} = y_n + (-y_n + e^{2x_n})\,h$$

$$y_1 = y_0 + (-y_0 + e^{2x_0})\,h$$

$$= 1 + (\underbrace{-1 + e^0}_{0}) \cdot .1$$

$$= 1$$

$$y_2 = y_1 + (-y_1 + e^{2x_1})\,h$$

$$= 1 + (-1 + e^{2(.1)}) \cdot .1$$

$$= 1.02214$$

Get less error if <u>you make h smaller</u> $h = .05$

k	x_k	y_k	Exact	Abs	Rel
0	0				
1	.05				
2	.10				
3	.15				
4	.20				
5	.25				
6	.30				
7	.35				
8	.40				
9	.45				
10	.50	1.28171	1.31045	.02874	.02193

2.2% error rate

Improved Euler's Method (Heun's Method)

Find the average slope between x_n & x_{n+1}

$$y_{n+1} = y_n + h\,\frac{f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})}{2}$$

$$y^*_{n+1} = y_n + h f(x_n, y_n)$$

Think of y^*_{n+1} as predicting $y(x_n)$ whereas y_{n+1} corrects it.

Improved Eulers

k	x_k	y^*_k	y_k	y_{exact}	Rel error
0					
1					
2					
3					
4					
5	0.5	1.25252	1.31979	1.31045	.00713

9.3 Three-Term Taylor Method

Use three terms from Taylor series

$$y(x_n+h) \approx \underbrace{y(x_n) + y'(x_n)h}_{2\text{ terms}} \Big\}\text{ Euler's Method}$$

So three terms is

$$y(x_n+h) \approx y(x_n) + y'(x_n)h + \frac{y''(x_n)h^2}{2}$$

So the three Term Taylor Method is

$$\boxed{y_{n+1} = y_n + f(x_n, y_n)h + \frac{1}{2} f'(x_n, y_n) h^2}$$

9.4 Runge Kutta Methods

4th order

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = h f(x_n, y_n)$$
$$k_2 = h f(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_1)$$
$$k_3 = h f(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_2)$$
$$k_4 = h f(x_n + h, y_n + k_3)$$