$\boxed{8.8}$ Variation of Parameters

Go back to end of 8.5

DEF: Fundamental Matrix.

Let $X_1 = \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix}$ $X_2 = \begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{pmatrix}, \dots, X_n = \begin{pmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{pmatrix}$

be a fundamental set of n solution vectors of the homogeneous system $X' = AX$ on an interval I.

The matrix

$$\Phi(t) = (X_1, \dots, X_n)$$

$$= \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ x_{n1} & \cdots & \cdots & x_{nn} \end{pmatrix}$$

Motivation behind def.

The fundamental set of solutions to the homogeneous system gives the general solution

$$X = c_1 X_1 + c_2 X_2 + \cdots + c_n X_n$$

$$\underbrace{\begin{pmatrix} \underset{n\times 1}{X_1} & \underset{n\times 1}{X_2} & \cdots & \underset{n\times 1}{X_n} \end{pmatrix}}_{n\times n} \underbrace{\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}}_{n\times 1}$$

$$X = \underset{n\times n}{\Phi(t)} \; \underset{n\times 1}{C}$$

Note: $X' = A(t)X$

$$= A(t)\Phi(t)C$$

But $X' = \Phi'(t)C$

So $X' = A(t)X$

$$\Phi'(t)C = A(t)\Phi(t)C$$

$$\Phi'(t)C - A(t)\Phi(t)C = 0$$

$$\left[\Phi'(t) - A(t)\Phi(t)\right]C = 0 \quad \text{(True for all } t\text{)}$$

$$\Phi'(t) - A(t)\Phi(t) = 0$$

$$\Phi'(t) = A(t)\Phi(t)$$

Note: $\Phi(t)$ is nonsingular!

DEF:

$$\Psi(t) = \Phi(t)\,\Phi^{-1}(t_0)$$

$$\Psi(t_0) = \Phi(t_0)\,\Phi^{-1}(t_0) = I$$

It can be shown $\Psi(t)$ is a fundamental matrix

$\boxed{\text{Section 8.8}}$ Solve $X' = AX + F$.

$X_c = \Phi(t)\,C$ is the solution to the homogeneous system.

So try a solution of the form

$$X_p = \Phi(t)\,U(t)$$

$$X_p' = \Phi(t)\,U'(t) + \Phi'(t)\,U(t)$$

Since $X_p' = AX_p + F$ and $X_p = \Phi(t)U(t)$, then

$$\overbrace{\Phi'(t)\,U(t) + \Phi(t)\,U'(t)}^{X_p'} = A\overbrace{\Phi(t)\,U(t)}^{X_p} + F(t)$$

$$A\,\cancel{\Phi(t)}\,U(t) + \Phi(t)\,U'(t) = A\,\cancel{\Phi(t)}\,U(t) + F(t)$$

(because $\Phi'(t) = A\,\Phi(t)$)

$$\Phi(t)\,U'(t) = F(t)$$

$$U'(t) = \Phi^{-1}(t)\,F$$

$$U(t) = \int \Phi^{-1}(t)\,F\,dt$$

Thus, $X_p = \Phi(t)\, u(t)$

$$= \Phi(t) \int \Phi^{-1}(t) F(t)\, dt$$

So it follows that

$$X = X_c + X_p$$

$$= \Phi(t) C + \Phi(t) \int \Phi^{-1}(t) F(t)\, dt$$

Ex: $X' = \begin{pmatrix} 0 & 2 \\ -1 & 3 \end{pmatrix} X + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^t$

$\lambda_1 = 1$, $\lambda_2 = 2$

$K_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $K_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Thus, the solution to the homogeneous is

$$X_c = c_1 X_1 + c_2 X_2$$

$$= c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$$

$$= \underbrace{\begin{bmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{bmatrix}}_{\Phi(t)} \underbrace{\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}}_{C}$$

For a 2x2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\underline{\Phi}^{-1}(t) = \frac{1}{|\underline{\Phi}(t)|}\begin{pmatrix} e^{2t} & -e^{2t} \\ -e^{t} & 2e^{t} \end{pmatrix} = \frac{1}{e^{3t}}\begin{pmatrix} e^{2t} & -e^{2t} \\ -e^{t} & 2e^{t} \end{pmatrix} = \begin{pmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{pmatrix}$$

$$X_p = \underline{\Phi}(t) \int \underline{\Phi}^{-1}(t)\, F(t)\, dt$$

So

$$\int \underline{\Phi}^{-1}(t)\, F(t)\, dt = \int \begin{pmatrix} e^{-t} & -e^{-t} \\ -e^{-2t} & 2e^{-2t} \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t}\, dt = \int \begin{pmatrix} 2e^{-t} \\ -3e^{-2t} \end{pmatrix} e^{t}$$

$$= \int \begin{pmatrix} 2 \\ -3e^{-t} \end{pmatrix} dt = \begin{bmatrix} 2t \\ 3e^{-t} \end{bmatrix}$$

$$X_p = \Phi(t)\int \Phi^{-1}(t)\,F(t)\,dt$$

$$= \begin{bmatrix} 2e^t & e^{2t} \\ e^t & e^{2t} \end{bmatrix}\begin{bmatrix} 2t \\ 3e^{-t} \end{bmatrix}$$

$$= \begin{pmatrix} 4te^t + 3e^t \\ 2te^t + 3e^t \end{pmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix} te^t + \begin{bmatrix} 3 \\ 3 \end{bmatrix} e^t$$

So the solution is

$$X = X_c + X_p = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^t + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{2t} + \begin{bmatrix} 4 \\ 2 \end{bmatrix}te^t + \begin{bmatrix} 3 \\ 3 \end{bmatrix}e^t$$