$\boxed{8.4}$ Matrices

Matrix Multip.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 4 & 7 & 10 \\ 5 & 8 & 11 \\ 6 & 9 & 12 \end{bmatrix} =$$

2×3 3×3

must match.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 5 \end{bmatrix}$$

3×2 3×2

Impossible

Dot products of the rows of the left matrix with the cols of the right matrix.
The result of the i^{th} row & j^{th} column is placed in the (i,j) entry in the new matrix.

dot product of two vectors

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \Rightarrow (1)(4) + (2)(5) + (3)(6)$$

$$4 + 10 + 18$$

$$= 32$$

Ex:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 4 & 7 & 10 \\ 5 & 8 & 11 \\ 6 & 9 & 12 \end{bmatrix} = \begin{bmatrix} \bigcirc & \bigcirc & \bigcirc \\ \bigcirc & 122 & 167 \end{bmatrix}$$

2×3 3×3

2nd row × 3rd column

$40 + 55 + 72 = 167$

2nd row × 2nd column

$28 + 40 + 54 = 122$

Multiplicative Identity

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Zero Matrix

$$O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Associative

$$A(BC) = (AB)C$$

Distributive

$$A(B+C) = AB + AC.$$

$$(B+C)A = BA + CA$$

Note: Commutative Law does not work for matrices

e.g. $AB \neq BC$.

Transpose

A^T = switch columns & rows.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \quad A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

Mult. Inverse

$$AB = BA = I$$

Non-singular $\Rightarrow$ Inverse of A exists.

Singular Matrix – no inverse exists.

Derivative of a Matrix of Funct

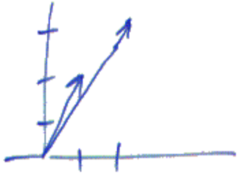
$$\frac{dA}{dt} = \left(\frac{d\,a_{ij}}{dt} \right)_{m \times n}$$

Integration of a Matrix

$$\int_{t_0}^{t} A(s)\,ds = \left(\int_{t_0}^{t} a_{ij}(s)\,ds \right)_{m \times n}$$

Eigenvalues & Eigenvalues

$x = \begin{bmatrix}1\\2\end{bmatrix} \Rightarrow$ vector $\Rightarrow$



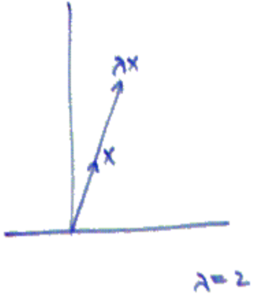
$A = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$

$Ax = \begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix}$

This matrix stretched & changed the direction of x.

Let A : nxn. λ is an eigenvalue and x is an eigenvector if

$$\underbrace{\underset{n\times n}{A}\,\underset{n\times 1}{x}}_{n\times 1} = \underset{1\times 1}{\lambda}\,\underset{n\times 1}{x}$$



A does not change the direction of x, but does change the length of x by a factor of λ.

However, if λ is negative, the direction is completely opposite of x

$\boxed{8.5}$ Matrices & Systems of Linear First Order Eq

$$X = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \qquad A = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1}(t) & \cdots & \cdots & a_{nn}(t) \end{pmatrix} \qquad F(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

When the system is homogeneous, then $F(t) = 0$ and

$$\boxed{X' = AX}$$

The system

$$\left.\begin{aligned} \frac{dx_1}{dt} &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t) \\ \frac{dx_2}{dt} &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + f_2(t) \\ &\vdots \\ \frac{dx_n}{dt} &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t) \end{aligned}\right\} \Rightarrow \text{can write as}$$

$$\boxed{X' = AX + F}$$

Ex:

$$\frac{dx}{dt} = -2x + 5y + e^t - 2t$$

$$\frac{dy}{dt} = 4x - 3y + 10t$$

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} -2 & 5 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e^t - 2t \\ 10t \end{bmatrix}$$

$$X' = A \quad X + F$$

$$\rightarrow F = \begin{pmatrix} e^t - 2t \\ 10t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + \begin{pmatrix} -2 \\ 10 \end{pmatrix} t$$

DEF: A solution vector on an interval I is any column vector

$$X = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

whose entries are differentiable functions satisfying the system $X' = AX + F$ on the int

EX:
Verify that $X_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-2t} = \begin{pmatrix} e^{-2t} \\ -e^{-2t} \end{pmatrix}$ and

$$X_2 = \begin{pmatrix} 3 \\ 5 \end{pmatrix} e^{6t} = \begin{pmatrix} 3e^{6t} \\ 5e^{6t} \end{pmatrix}$$

are solutions of $X' = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} X$

$$X_1' = \begin{pmatrix} -2e^{-2t} \\ 2e^{-2t} \end{pmatrix} \qquad AX_1 = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} e^{-2t}$$

they match!

$$X_2' = \begin{pmatrix} 18e^{6t} \\ 30e^{6t} \end{pmatrix} \qquad AX_2 = \begin{bmatrix} 1 & 3 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} e^{6t} = \begin{bmatrix} 18 \\ 30 \end{bmatrix} e^{6t}$$

they match!

We're skipping the fundamental matrix in 8.5. We will return to it in sec 8.8.