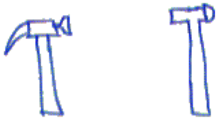
$\boxed{7.6}$ Dirac Delta Function

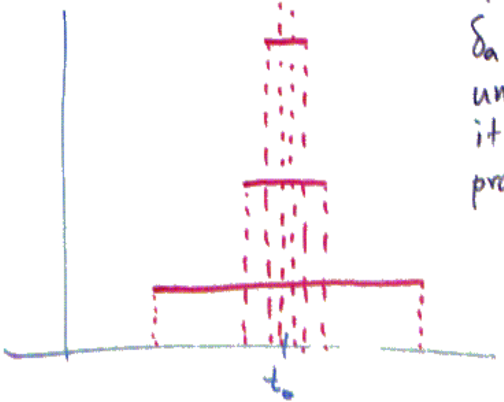
Unit Impulse

Mechanical systems are often acted upon by an external force of large magnitude that acts only for a short period of time:

- lightning
- sharp blow of a ball peen hammer.
- baseball hit by a bat.



$$\delta_a(t-t_0) = \begin{cases} 0, & 0 \le t < t_0 - a \\ \frac{1}{2a} & t_0 - a \le t < t_0 + a \\ 0, & t \ge t_0 + a \end{cases} \qquad \begin{matrix} a > 0 \\ t_0 > 0 \end{matrix}$$



behavior of δ_a as $a \to 0$

The function $\delta_a(t-t_0)$ is called unit impulse since it possesses the property

$$\int_0^\infty \delta_a(t-t_0)\,dt = 1$$

Dirac Delta Function

Def. (Intuitive def)

$$\delta(t-t_0) = \lim_{a \to 0} \delta_a(t-t_0)$$

Note: $\delta(t-t_0)$ is not a function since its value at t_0 is ∞!

(i) $\delta(t-t_0) = \begin{cases} \infty, & t = t_0 \\ 0, & t \neq t_0 \end{cases}$

(ii) $\int_0^\infty \delta(t-t_0)\,dt = 1$

Further, it is not of exponential order. However, the Laplace transform still exists.

Does not contradict Thm 7.1!

Read the remarks at the end of the section.

This "function" was met with skeptism since it does not behave like a regular function. It still produces correct results. It was place with a much better definition of

$$\int_0^\infty f(t)\,\delta(t-t_0)\,dt = f(t_0) \qquad (*)$$

It "sifts" the function $f(t)$ and produces the value $f(t_0)$

Laplace transform

$$\mathcal{L}\{\delta(t-t_0)\} = \int_0^\infty e^{-st}\delta(t-t_0)\,dt = e^{-st_0}$$

Note therefore that $\mathcal{L}\{\delta(t)\} = e^{-s(0)} = 1$

So $\mathcal{L}\{\delta(t-t_0)\} = e^{-st_0}$

$$\mathcal{L}\{\delta(t)\} = 1$$

EX: $y'' + 2y' = \delta(t-1)$ $\quad y(0)=0$, $y'(0)=1$

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} = \mathcal{L}\{\delta(t-1)\}$$

$$s^2Y(s) - s\cancel{y(0)} - \underbrace{y'(0)}_{1} + 2[sY(s) - \cancel{y(0)}] = e^{-s}$$

$$(s^2+2s)Y(s) = 1 + e^{-s}$$

$$Y(s) = \frac{1+e^{-s}}{s^2+2s} = \frac{1+e^{-s}}{s^2+2s+1-1}$$

$$= \frac{1+e^{-s}}{(s+1)^2-1} = \frac{1}{(s+1)^2-1} + \frac{e^{-s}}{(s+1)^2-1}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2-1}\right\} + \mathcal{L}^{-1}\left\{\frac{e^{-s}}{(s+1)^2-1}\right\} \to a=1 \Rightarrow f(t-1)\mathcal{U}(t-1)$$

$$= e^{-t}\sinh(t) + e^{-(t-1)}\sinh(t-1)\,\mathcal{U}(t-1)$$

Start with

$$Y(s) = \frac{1+e^{-s}}{s(s+2)} = \frac{\frac{1}{2}(1+e^{-s})}{s} + \frac{-\frac{1}{2}(1+e^{-s})}{s+2}$$

$$\frac{1}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2} \Rightarrow 1 = A(s+2) + Bs$$

$$= (A+B)s + 2A$$

$$A = \tfrac{1}{2} \Rightarrow B = -\tfrac{1}{2}$$

$$Y(s) = \frac{\frac{1}{2}}{s} - \frac{\frac{1}{2}}{s+2} + \frac{1}{2}\frac{e^{-s}}{s} - \frac{1}{2}\frac{e^{-s}}{s+2}$$

$$y(t) = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s+2}\right\}$$

$$= \frac{1}{2} - \frac{e^{-2t}}{2} + \frac{1}{2}\mathcal{U}(t-1) - \frac{1}{2}e^{-2(t-1)}\mathcal{U}(t-1)$$

$$= e^{-t}\underbrace{\left[\frac{e^{t}-e^{-t}}{2}\right]}_{\sinh(t)} + \mathcal{U}(t-1)e^{-(t-1)}\underbrace{\left[\frac{e^{t-1}-e^{-(t-1)}}{2}\right]}_{\sinh(t-1)}$$

$$= e^{-t}\sinh(t) + e^{-(t-1)}\sinh(t-1)\,\mathcal{U}(t-1)$$