7.5 Applications of the Laplace Transf.

Solving a diff-eq.

$$a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = g(t)$$

$$y(0) = y_0,\ y'(0) = y_0',\ \cdots,\ y^{(n-1)}(0) = y_0^{(n-1)}$$

$$(*) \quad a_n \mathcal{L}\left\{\frac{d^n y}{dt^n}\right\} + a_{n-1}\mathcal{L}\left\{\frac{d^{n-1}y}{dt^{n-1}}\right\} + \cdots + a_1 \mathcal{L}\left\{\frac{dy}{dt}\right\} + a_0 \mathcal{L}\{y\} = \mathcal{L}\{g(t)\}$$

Since $\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$

Then (*) becomes

$$[a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0]\,Y(s) = a_n[s^{n-1}y_0 + \cdots + y_0^{(n-1)}] + a_{n-1}[s^{n-2}y_0 + \cdots + y_0^{(n-2)}] + \cdots + G(s)$$

$$Y(s) = \frac{}{} \Rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$Y(s) = \mathcal{L}\{y(t)\}$

$G(s) = \mathcal{L}\{g(t)\}$

Ex: $y'' + 2y' + y = e^{5t}$ $\quad y(0)=1,\ y'(0)=2.$

$$\mathcal{L}\{y''\} = s^2 Y(s) - s y(0) - y'(0) = s^2 Y(s) - s - 2$$

$$2\mathcal{L}\{y'\} = 2(sY(s) - y(0)) = 2(sY(s) - 1)$$

$$\mathcal{L}\{y\} = Y(s) = Y(s)$$

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = [s^2 + 2s + 1]Y(s) - s - 2 - 2 = \mathcal{L}\{e^{5t}\}$$

$$\Rightarrow \underbrace{[s^2 + 2s + 1]}_{(s+1)^2} Y(s) = s + 4 + \frac{1}{s-5} = \frac{(s-5)(s+4)+1}{s-5} = \frac{s^2 - s - 19}{s-5}$$

$$Y(s) = \frac{s^2 - s - 19}{(s+1)^2(s-5)}$$

$$\frac{s^2 - s - 19}{(s+1)^2(s-5)} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s-5}$$

$$s^2 - s - 19 = (A+C)s^2 + (-4A + B + 2C)s + (-5A - 5B + C)$$

$$\begin{aligned} A + C &= 1 \\ -4A + B + 2C &= -1 \\ -5A - 5B + C &= -19 \end{aligned} \Rightarrow \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ -4 & 1 & 2 \\ -5 & -5 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} A \\ B \\ C \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ -1 \\ -19 \end{bmatrix}}_{b}$$

$$x = A^{-1}b = \begin{bmatrix} 35/36 \\ 17/6 \\ 1/36 \end{bmatrix}$$

So $Y(s) = \frac{35/36}{s+1} + \frac{17/6}{(s+1)^2} + \frac{1/36}{s-5}$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{35/36}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{17/6}{(s+1)^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1/36}{s-5}\right\}$$

$$\boxed{y(t) = \frac{35}{36}e^{-t} + \frac{17}{6}te^{-t} + \frac{1}{36}e^{5t}}$$

Ex: $y'' + 2y' - 3y = \sin 2t$ $\qquad y(0) = 1 = y_0$, $y'(0) = 2 = y_0'$

$$\mathcal{L}\{y''\} = s^2Y(s) - sy_0 - y_0'$$

$$\mathcal{L}\{y'\} = sY(s) - y_0$$

$$\mathcal{L}\{y'' + 2y' - 3y\} = s^2Y(s) - sy_0 - y_0' + 2sY(s) - 2y_0 - 3Y(s)$$

$$= [s^2 + 2s - 3]Y(s) - sy_0 - 2y_0 - y_0' = \frac{2}{s^2+4}$$

$$(s+3)(s-1)Y(s) = \frac{2}{s^2+4} + \frac{(sy_0 + \overbrace{2y_0 + y_0'}^{b})(s^2+4)}{s^2+4}$$

$$Y(s) = \frac{2 + (sy_0 + b)(s^2+4)}{(s+3)(s-1)(s^2+4)} = \frac{2 + (s^3y_0 + 4sy_0 + bs^2 + 4b)}{(s+3)(s-1)(s^2+4)}$$

$$= \frac{s^3y_0 + bs^2 + 4sy_0 + (4b+2)}{(s+3)(s-1)(s^2+4)}$$

$$\frac{y_0 s^3 + bs^2 + 4y_0 s + (4b+2)}{(s+3)(s-1)(s^2+4)} = \frac{A}{s-1} + \frac{B}{s+3} + \frac{Cs+D}{s^2+4}$$

$\therefore$ lotsa algebra.

$$y_0 s^3 + bs^2 + 4y_0 s + (4b+2) = (A+B+C)s^3 + (3A-B+2C+D)s^2 + (4A+4B-3C+2D)s + (12A-4B-3D)$$

$$A+B+C = y_0$$

$$3A-B+2C+D = b$$

$$4A+4B-3C+2D = 4y_0$$

$$12A-4B-3D = 4b+2$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 3 & -1 & 2 & 1 \\ 4 & 4 & -3 & 2 \\ 12 & -4 & 0 & -3 \end{bmatrix} \quad x = \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} \quad b = \begin{bmatrix} y_0 \\ b \\ 4y_0 \\ 4b+2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1/20 & 1/20 & 1/20 & 1/20 \\ 27/52 & -9/52 & 3/52 & -1/52 \\ 28/65 & 8/65 & 7/65 & -2/65 \\ -32/65 & 28/65 & 8/65 & -7/65 \end{bmatrix}$$