$$= \frac{-2s(s^2-4)\left[-s^2-12\right]}{(s^2-4)^4}$$

$$= \frac{2s(s^2+12)}{(s^2-4)^3}$$

$$\mathcal{L}\{t^2 e^{3t}\cosh 2t\} = \left[\frac{2s(s^2+12)}{(s^2-4)^3}\right]_{s \to s-3}$$

$$= \frac{2(s-3)\left((s-3)^2+12\right)}{\left((s-3)^2-4\right)^3}$$

$\boxed{7.4}$ Transforms of Derivatives, Integrals, and of Periodic Functions.

We want to use the Laplace Transform to solve differential equations. To do that, we need to evaluate transforms of derivs.

$y = f(t)$

$y' = f'(t)$

$$\mathcal{L}\{f'(t)\} = \int_0^\infty \underbrace{e^{-st}}_{u} \underbrace{f'(t)\,dt}_{dv}$$

$u = e^{-st}$ $\quad dv = f'(t)\,dt$

$du = -se^{-st}\,dt$ $\quad v = f(t)$

$$= \left[e^{-st} f(t)\right]_0^\infty + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= \underbrace{\lim_{b\to\infty} \frac{f(b)}{e^{bs}}}_{\text{If } f(b) \text{ is of exponential order, then } \lim_{b\to\infty} \sim \,=\, 0} - f(0) + s\mathcal{L}\{f(t)\}$$

$$\mathcal{L}\{f'(t)\} = -f(0) + sF(s)$$

Similarly,

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

In general,

If $f(t), f'(t), \dots, f^{(n-1)}(t)$ are continuous on $[0,\infty)$ and are of exponential order, and if $f^{(n)}(t)$ is piecewise cont on $[0,\infty)$

$$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

Ex: Note: $\frac{d}{dt} te^{5t} = 5te^{5t} + e^{5t}$

$f(0) = 0$
$f'(0) = 0$
all are zero that we need for this example

So

$$\mathcal{L}\{5te^{5t} + e^{5t}\} = \mathcal{L}\left\{\frac{d}{dt} te^{5t}\right\}$$

$$= s\mathcal{L}\{te^{5t}\}$$

$$= s(-1)^1\frac{d}{ds}\left(\frac{1}{s-5}\right)$$

$$= -s(-1)(s-5)^{-2}$$

$$= \frac{s}{(s-5)^2}$$

Ex: Note: $\frac{d}{dt}\, te^{5t} = 5te^{5t} + te^{5t}$

$$\mathcal{L}\{5te^{5t} + e^{5t}\} = \mathcal{L}\left\{\frac{d}{dt}(te^{5t})\right\}$$

$$= sF(s) - f(0)$$

$$= s\mathcal{L}\{te^{5t}\} - 0$$

$$= s\,(-1)^1 \frac{d}{ds}\left(\frac{1}{s-5}\right)$$

$$= -s\left(\frac{-1}{(s-5)^2}\right)$$

$$= \frac{s}{(s-5)^2}$$

Convolution

If function f & g are piecewise cont. on $[0,\infty)$, then the convolution of f & g denote by $f*g$, is given by

$$f*g = \int_0^t f(z)\,g(t-z)\,dz$$

Note: $f*g = g*f$ (prove it!)

Ex: $\underset{f}{t^2} * \underset{g}{e^t}$

$$= \int_0^t f(z)\,g(t-z)\,dz$$

$$= \int_0^t z^2 e^{t-z}\,dz$$

$$= e^t \int_0^t z^2 e^{-z}\,dz$$

$$= e^t\left[-z^2e^{-z} - 2ze^{-z} - 2e^{-z}\right]_0^t$$

$$= e^t\left[-t^2e^{-t} - 2te^{-t} - 2e^{-t}\right] + 2e^t$$

$$= -t^2 - 2t - 2 + 2e^t$$

$$\boxed{\begin{array}{ccc} z^2 & + & e^{-z} \\ 2z & - & -e^{-z} \\ 2 & + & e^{-z} \\ 0 & & -e^{-z} \end{array}}$$

Is it possible to find $\mathcal{L}\{t^2 * e^t\}$ without finding $t^2 * e^t$?

$\boxed{\text{Thm 7.9}}$ Let $f(t)$ & $g(t)$ be piecewise cont on $[0,\infty)$ and of exponential order; then

$$\mathcal{L}\{f*g\} = \mathcal{L}\{f\}\mathcal{L}\{g\}$$
$$= F(s)\,G(s)$$

Proof: $F(s) = \mathcal{L}\{f\} = \int_0^\infty e^{-sw} f(w)\,dw$

$G(s) = \mathcal{L}\{g\} = \int_0^\infty e^{-sy} g(y)\,dy$

$$F(s)G(s) = \int_0^\infty e^{-sw} f(w)\,dw \int_0^\infty e^{-sy} g(y)\,dy$$

$$= \int_0^\infty \int_0^\infty e^{-s(w+y)} f(w)\,g(y)\,dy\,dw$$

$t = w+y$ (w is const wrt y)

$dt = dy$

$$= \int_0^\infty \int_w^\infty e^{-st} f(w)\,g(t-w)\,dt\,dw$$

Interchange the order of integration (we can do this since f & g are piecewise cont on $[0,\infty)$ and of exponential order.

$0 < w < t < \infty \Rightarrow 0 < w < t$ and $0 < t < \infty$

$$= \int_0^\infty \int_0^t e^{-st} f(w)\,g(t-w)\,dw\,dt$$

$$= \int_0^\infty e^{-st} \underbrace{\left\{ \int_0^t f(w)\, g(t-w)\, dw \right\}}_{f * g} dt$$

$$= \int_0^\infty e^{-st} (f * g)\, dt$$

$$= \mathcal{L}\{ f * g \}$$

Corollary:

$$\mathcal{L}\left\{ \int_0^t f(w)\, dw \right\} = \mathcal{L}\{ f * 1 \} = \mathcal{L}\{f\}\, \mathcal{L}\{1\}$$

$$= \frac{F(s)}{s}$$

Ex: $\mathcal{L}^{-1}\left\{ \frac{1}{(s-1)(s+4)} \right\} = \mathcal{L}^{-1}\{ \underbrace{F(s)}_{\frac{1}{s-1}} \cdot \underbrace{G(s)}_{\frac{1}{s+4}} \}$

$$= f * g \qquad f = e^{t} \quad g = e^{-4t}$$

$$= \int_0^t f(w)\, g(t-w)\, dw = \int_0^t e^{w} e^{-4(t-w)}\, dw$$

$$= e^{-4t} \int_0^t e^{5w}\, dw = e^{-4t} \left. \frac{e^{5w}}{5} \right|_0^t$$

$$= e^{-4t} \left(\frac{e^{5t}}{5} - \frac{1}{5} \right) = \frac{1}{5} e^{t} - \frac{1}{5} e^{-4t}$$

Transform of a Periodic function

Let $f(t)$ be piecewise cont on $[0,\infty)$ and of exponential order. If $f(t)$ is periodic with period T, then

$$\mathcal{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

Proof:

$$\mathcal{L}\{f\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^T e^{-st} f(t)\,dt + \int_T^\infty e^{-st} f(t)\,dt$$

$t = w + T$

$dt = dw$

$$= \int_0^T e^{-st} f(t)\,dt + \int_0^\infty e^{-s(w+T)} \underbrace{f(w+T)}_{f(w)}\,dw$$

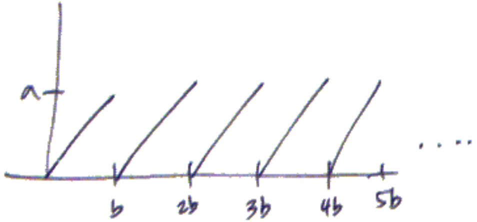
$$= \int_0^T e^{-st} f(t)\,dt + e^{-sT} \underbrace{\int_0^\infty e^{-sw} f(w)\,dw}_{\mathcal{L}\{f\}}$$

$$\mathcal{L}\{f\} = \int_0^T e^{-st} f(t)\,dt + e^{-sT}\mathcal{L}\{f\}$$

$$(1-e^{-sT})\mathcal{L}\{f\} = \int_0^T e^{-st} f(t)\,dt$$

$$\mathcal{L}\{f\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

Periodic function



Period $= b$

$$\mathcal{L}\{f\} = \frac{a/b}{1-e^{-sb}} \int_0^b t\, e^{-st}\, dt$$

$t \quad e^{-st}$

$1 \quad -\frac{1}{s} e^{-st}$

$0 \quad \frac{1}{s^2} e^{-st}$

$$= \frac{a/b}{1-e^{-sb}} \left[-\frac{1}{s} t e^{-st} - \frac{1}{s^2} e^{-st} \right]_0^b$$

$$= \frac{a/b}{1-e^{-sb}} \left[-\frac{1}{s} b e^{-bs} - \frac{1}{s^2} e^{-bs} + \frac{1}{s^2} \right]$$

$$= \frac{a}{bs^2(1-e^{-sb})} \left[-sb e^{-bs} - e^{-bs} + 1 \right]$$

$$= \frac{a}{s} \left(\frac{1}{sb} + \frac{1}{1-e^{-sb}} \right)$$

Ex: $\mathcal{L}\{\cos 2t\} = \dfrac{s}{s^2+4}$

$$\int e^{-st}\cos 2t\,dt = \frac{1}{2}e^{-st}\sin 2t - \frac{1}{4}s e^{-st}\cos 2t - \frac{s^2}{4}\int e^{-st}\cos 2t\,dt$$

$$\boxed{\begin{array}{lcl} e^{-st} & + & \cos 2t \\ -se^{-st} & - & \frac{\sin 2t}{2} \\ s^2 e^{-st} & + & \frac{-\cos 2t}{4} \end{array}}$$

$$\left(1+\frac{s^2}{4}\right)\int e^{-st}\cos 2t\,dt = \frac{1}{4}e^{-st}(2\sin 2t - s\cos 2t)$$

$$\int e^{-st}\cos 2t\,dt = \frac{4}{4+s^2}\,\frac{1}{4}e^{-st}(2\sin 2t - s\cos 2t)$$

$$= \frac{1}{4+s^2}e^{-st}(2\sin 2t - s\cos 2t)$$

$$\mathcal{L}\{\cos 2t\} = \frac{1}{1-e^{-s\pi}} \int_0^{\pi} e^{-st} \cos 2t \, dt$$

$$= \frac{1}{1-e^{-s\pi}} \frac{1}{4+s^2} \Big[e^{-st} (2\sin 2t - s\cos 2t) \Big]_0^{\pi}$$

$$= \frac{1}{1-e^{-s\pi}} \frac{1}{4+s^2} \Big[e^{-s\pi}(-s) - e^{0}(-s) \Big]$$

$$= \frac{1}{\cancel{1-e^{-s\pi}}} \frac{1}{4+s^2} s \Big[\cancel{1 - e^{-s\pi}} \Big]$$

$$= \frac{s}{s^2+4}$$