$\boxed{7.1}$ Laplace Transform

Applying the Laplace Transform to a linear n^{th} order diff-eq

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = g(t)$$

changes it to an algebraic equation that involves the conditions

$$y(0), y'(0), \ldots, y^{(n-1)}(0)$$

Transform

$$\int x^2 \, dx = \frac{x^3}{3} + c$$

Transforms x^2 to $\frac{x^3}{3} + c$

Note that differentiation & integration are both linear operators

$$\int \alpha f + \beta g \, dx = \alpha \int f dx + \beta \int g \, dx$$

We want to do an integral transform

$$\int_a^b K(s,t)\, f(t)\, dt$$ that transforms

$f(t)$ into a function of s.

E.g. $f(t) \longrightarrow g(s)$

Basic Definition

If $f(t)$ is defined for $t \geq 0$, then

$$\int_0^\infty K(s,t)\, f(t)\, dt = \lim_{b\to\infty} \int_0^b K(s,t)\, f(t)\, dt$$

$\boxed{\text{DEF 7.1}}$ Let f be a function defined for $t \geq 0$.

Then the integral

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

is said to be the Laplace transform of f provided the integral converges.

Notation

$$\mathcal{L}\{f(t)\} = F(s)$$

$$\mathcal{L}\{g(t)\} = G(s)$$

Ex: Evaluate $\mathcal{L}\{1\}$

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}(1)\,dt = \left.\frac{e^{-st}}{-s}\right|_0^\infty = 0 - \left(-\frac{1}{s}\right) = \frac{1}{s}, \quad s>0.$$

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathcal{L}\{f\} + \beta \mathcal{L}\{g\}$$

$\rightarrow$ Laplace transform is linear!

Exponential Order

A function f is said to be of exponential order if there exist numbers c, $M>0$, and $T>0$ such that

$$|f(t)| \leq Me^{ct} \quad \text{for } t>T$$

Sufficient conditions for the existence of the Laplace transforms.

Let $f(t)$ be piecewise cont. on the interval $[0,\infty)$ and is of exponential order for $t>T$, then $\mathcal{L}\{f(t)\}$ exists for $s>c$.

$$\mathcal{L}\{1\} = \frac{1}{s}$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\mathcal{L}\{\sin kt\} = \frac{k}{s^2+k^2}$$

$$\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}$$

$$\mathcal{L}\{\cosh kt\} = \frac{s}{s^2-k^2}$$

$$\mathcal{L}\{\sinh kt\} = \frac{k}{s^2-k^2}$$

$$\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = \left.\frac{e^{-(s-a)t}}{-(s-a)}\right|_0^\infty, \quad s-a>0,\ s>a$$

$$= 0 - \left(\frac{1}{-(s-a)}\right)$$

$$= \frac{1}{s-a}$$

$\boxed{7.1}$ Cont.

$$\mathcal{L}\{\cos kt\} = \int_0^\infty e^{-st}\cos kt\,dt = \left[\frac{1}{k}e^{-st}\sin kt - \frac{s}{k^2}e^{-st}\cos kt\right]_0^\infty - \int_0^\infty \frac{s^2}{k^2}e^{-st}\cos kt\,dt$$

$$\begin{array}{lcl} e^{-st} & + & \cos kt \\ -se^{-st} & - & \dfrac{\sin kt}{k} \\ s^2e^{-st} & + & \dfrac{-\cos kt}{k^2} \end{array}$$

$$\left(1+\frac{s^2}{k^2}\right)\underbrace{\int_0^\infty e^{-st}\cos kt\,dt}_{\mathcal{L}\{\cos kt\}} = \left[\frac{1}{k}e^{-st}\sin kt - \frac{s}{k^2}e^{-st}\cos kt\right]_0^\infty$$

$$\left(\frac{k^2+s^2}{k^2}\right)\mathcal{L}\{\cos kt\} = \frac{s}{k^2}$$

$$\mathcal{L}\{\cos kt\} = \frac{k^2}{k^2+s^2}\left(\frac{s}{k^2}\right) = \frac{s}{k^2+s^2}$$