$\boxed{6.3}$ Solutions about ordinary points

Suppose $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$
is divided by $a_2(x)$.

$$y'' + P(x)y' + Q(x)y = 0$$

DEF: A point x_0 is said to be an ordinary point of the differential equation if both $P(x)$ and $Q(x)$ are analytic at x_0.

(Analytic $\rightarrow$ it can be represented by a power series with positive radius of convergence.)

A point that is not an ordinary pt is said to be a singular pt of the equation.

Ex: $y'' + e^x y + x^2 y = 0$

any finite x is an ordinary pt since e^x & x^2 converge for all finite values of x.

Ex: The diff-eq

$$y'' + (\ln x)y = 0$$

has a singular pt at $x=0$ since $Q(x) = \ln x$ possesses no power series centered at 0

Ex: $(x^2-1)y'' + 2xy' + 6y = 0$

Singular pts at $x = \pm 1$

ordinary pts $\Rightarrow$ all other finite values of x.

$\boxed{\text{Thm } 6.1}$ Existence of a Power Series Solution

If $x = x_0$ is an ordinary pt of the diff-eq $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$, then we can always find two linearly independent solutions in the form of power series centered at x_0.

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

A series solution converges at least for $|x - x_0| < R$, where R is the distance from x_0 to the closest singular pt.

Ex: $y'' - 2xy = 0$

Soln: No singular pts $\Rightarrow$ converges for finite x

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

$$0 = y'' - 2xy$$

$$= \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - 2x \sum_{n=0}^{\infty} c_n x^n = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} - \sum_{n=0}^{\infty} 2 c_n x^{n+1}$$

$k = n-2$, $n = k+2$ $\quad$ $k = n+1$, $n = k-1$

$$= \sum_{k=0}^{\infty} (k+2)(k+1) c_{k+2} x^k - \sum_{k=1}^{\infty} 2 c_{k-1} x^k$$

$$= 2c_2 + \sum_{k=1}^{\infty} \left[(k+2)(k+1) c_{k+2} - 2 c_{k-1} \right] x^k$$

$$2c_2 = 0 \Rightarrow c_2 = 0$$

$$(k+2)(k+1) c_{k+2} - 2 c_{k-1} = 0$$

$$(k+2)(k+1)\,c_{k+2} = 2c_{k-1}$$

$$c_{k+2} = \frac{2c_{k-1}}{(k+2)(k+1)}, \quad k \geq 1$$

$n = k+2 \Rightarrow k = n-2.$

$$\boxed{c_n = \frac{2c_{n-3}}{n(n-1)}}, \quad n \geq 3$$

$$c_3 = \frac{2c_0}{3(2)}$$

$$c_4 = \frac{2c_1}{4\,3}$$

$$c_5 = \frac{2c_2}{5\cdot 4} = 0$$

$$c_6 = \frac{2c_3}{6\cdot 5} = \frac{2}{6\cdot 5}\left(\frac{2c_0}{3\cdot 2}\right) = \frac{2^2 c_0}{6\cdot 5\cdot 3\cdot 2} = \frac{2^2\cdot 4c_0}{6!}$$

$$c_7 = \frac{2c_4}{7\cdot 6} = \frac{2}{7\cdot 6}\left(\frac{2c_1}{4\cdot 3}\right)\frac{5}{5}\cdot\frac{2}{2}\cdot \quad \frac{2^2\cdot 5\cdot 2c_1}{7!}$$

$$c_8 = \frac{2c_5}{8\cdot 7} = 0$$

$$c_9 = \frac{2c_6}{9\cdot 8} = \frac{2}{9\cdot 8}\left(\frac{2^2\cdot 4\cdot c_0}{6!}\right)\frac{7}{7} = \frac{2^3\cdot 4\,7\,c_0}{9!}$$

$$c_{10} = \frac{2c_7}{10\cdot 9} = \frac{2^3\cdot 8\cdot 5\cdot 2c_1}{10!}$$

$$c_{11} = \frac{2c_8}{11\cdot 10} = 0$$

Thus, the solution is

$$y = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{3n} x^{3n} + \sum_{n=0}^{\infty} c_{3n+1} x^{3n+1} + \sum_{n=0}^{\infty} c_{3n+2} x^{3n+2}$$

$$= c_0 \sum_{n=0}^{\infty} \frac{2^n \cdot (3n-2)\cdot(3n-5)\cdots 1}{(3n)!} x^{3n}$$

$$+ c_1 \sum_{n=0}^{\infty} \frac{2^n (3n-1)(3n-4)\cdots 2}{(3n+1)!} x^{3n+1}$$

$$+ \sum_{n=0}^{\infty} 0 \cdot x^{3n+2}$$

$$= c_0 y_0 + c_1 y_1$$

Two linearly independent solutions.