$\boxed{6.1}$ Cauch-Euler Equation

Any linear diff-eq of the form

these match!

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = g(x)$$

Case: Quadratic Case

$$a x^2 \frac{d^2 y}{dx^2} + b x \frac{dy}{dx} + c y = 0$$

$I = (0, \infty)$ since $\frac{d^2y}{dx^2}$ coeff is 0 at $x=0$

Try solution of the form

$$y = x^m$$

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$$y' = m x^{m-1}$$

$$y'' = m(m-1) x^{m-2}$$

So

$$a x^2 m(m-1) x^{m-2}$$

$$+ b x m x^{m-1}$$

$$+ c x^m$$

$$= x^m \left(am(m-1) + bm + c\right) = 0$$

Thus

$$am(m-1)+bm+c=0$$

$$am^2+(b-a)m+c=0$$

$$m=\frac{-(b-a)\pm\sqrt{(b-a)^2-4ac}}{2a}=m_1,m_2$$

Three cases:

Two real roots
one real root
two complex roots.

$\boxed{\text{Case I}}$ Two real roots.

The solution is $y=c_1x^{m_1}+c_2x^{m_2}$

$\boxed{\text{Case II}}$ Repeated root

Suppose the root is $m_1 \Rightarrow y_1=x^{m_1}$ is a solution.

$$y_2=y_1\int\frac{e^{-\int P dx}}{y_1^2}dx \qquad -P(x)=-\frac{b}{ax}\Rightarrow e^{-\int P(x)dx}=e^{-\frac{b}{a}\ln|x|}=x^{-\frac{b}{a}}$$

$$=x^{m_1}\int\frac{x^{-\frac{b}{a}}}{(x^{m_1})^2}dx$$

If m_1 is the only root, then

$$(b-a)^2=4ac$$

$$\boxed{m_1=\frac{-(b-a)}{2a}}$$

$$y_2 = x^{m_1}\int \frac{x^{-\frac{b}{a}}}{x^{2m_1}}\,dx = x^{m_1}\int \frac{\cancel{x^{-\frac{b}{a}}}}{x^{-\frac{(b-a)}{a}}}\,dx = x^{m_1}\int \frac{1}{x}\,dx = x^{m_1}\ln x$$

$$\boxed{2m_1 = \frac{-(b-a)}{a}}$$

Thus, the general solution is $y = c_1 x^{m_1} + c_2 x^{m_1}\ln x$

Note:

$$x^{\beta i} = e^{\ln x^{\beta i}} = e^{i(\beta \ln x)}$$

$$= \cos(\beta \ln x) + i\sin(\beta \ln x)$$

$$x^{-\beta i} = \cos(\beta \ln x) - i\sin(\beta \ln x)$$

$\boxed{\text{Case } \underline{III}}$ Complex roots. (Conjugate pairs)

Two roots: $\alpha \pm \beta i$

$$y = c_1 x^{\alpha+\beta i} + c_2 x^{\alpha-\beta i}$$

$$= x^{\alpha}\left(c_1 x^{\beta i} + c_2 x^{-\beta i}\right)$$

$$y = x^{\alpha}\left[c_1 x^{\beta i} + c_2 x^{-\beta i}\right]$$

$$= x^{\alpha}\left[c_1\{\cos(\beta \ln x) + i\sin(\beta \ln x)\} + c_2\left[\cos(\beta \ln x) - i\sin(\beta \ln x)\right]\right]$$

$$= x^{\alpha}\left[\underbrace{(c_1 + c_2)}\cos(\beta \ln x) + \underbrace{i(c_1 - c_2)}_{\text{any constant will do}}\sin(\beta \ln x)\right]$$

$$= x^{\alpha}\left(c_1^{*}\cos(\beta \ln x) + c_2^{*}\sin(\beta \ln x)\right)$$

Alternative method:

substitute $x = e^t$ or $t = \ln x$ and do the stuff from 4.6.

Ex: Solve $3x^2y'' + 6xy' + y = 0$

Method 1 $\quad y = x^m,\ y' = mx^{m-1},\ y'' = m(m-1)x^{m-2}$

$$3x^2\left(m(m-1)x^{m-2}\right) + 6x\left(mx^{m-1}\right) + x^m = 0$$

$$x^m\left[3m(m-1) + 6m + 1\right] = 0$$

$$x^m\left[3m^2 + 3m + 1\right] = 0$$

$$m = \frac{-3 \pm \sqrt{9 - 4(3)}}{6} = \frac{-3 \pm i\sqrt{3}}{6}$$

$\alpha = -\frac{1}{2} \quad \beta = \frac{\sqrt{3}}{6}$

$$y_1 = x^{-\frac{1}{2}}\left[c_1\cos\left(\frac{\sqrt{3}}{6}\ln x\right) + c_2\sin\left(\frac{\sqrt{3}}{6}\ln x\right)\right]$$

$\boxed{\text{Method 2}}$ $\quad x = e^t, \quad t = \ln x$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt}\cdot\frac{1}{x}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = \frac{1}{x}\,\frac{d}{dx}\left(\frac{dy}{dt}\right) + \frac{d}{dx}\left(\frac{1}{x}\right)\frac{dy}{dt}$$

$$= \frac{1}{x}\,\frac{d^2y}{dt^2}\underbrace{\frac{dt}{dx}}_{\frac{1}{x}} - \frac{1}{x^2}\frac{dy}{dt}$$

$$= \frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)$$

$$3x^2y'' + 6xy' + y = 0$$

$$3x^2\left(\frac{1}{x^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right)\right) + 6x\left(\frac{1}{x}\frac{dy}{dt}\right) + y = 0$$

$$3\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 6\frac{dy}{dt} + y = 0$$

$$3\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + y = 0$$

Solve $3m^2 + 3m + 1 = 0 \Rightarrow m = -\frac{1}{2} \pm \frac{\sqrt{3}}{6}i$

$$y = c_1 e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{3}}{6}t\right) + c_2 e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{6}t\right)$$

$$= c_1 x^{-1/2}\cos\left(\frac{\sqrt{3}}{6}\ln x\right) + c_2 x^{-1/2}\sin\left(\frac{\sqrt{3}}{6}\ln x\right)$$