$\boxed{5.3}$ Cont.

Ex: Interpret & solve the IVP

$$\frac{1}{5}\frac{d^2x}{dt^2} + 1.2\frac{dx}{dt} + 2x = 5\cos 4t$$

$\underbrace{x(0) = 1}_{\text{starts 1 unit below equilibrium}}$, $\underbrace{x'(0) = 0}_{\text{starts at rest}}$

$m = \frac{1}{5}$ slug or kg

$k = 2 \Rightarrow$ spring constant

$\beta = 1.2$ (Motion is damped)

$5\cos 4t \rightarrow$ External force w/ period $\pi/2$

Solve: $\dfrac{d^2x}{dt^2} + 6\dfrac{dx}{dt} + 10x = 25\cos 4t$

$(D^2 + 6D + 10)x = 25\cos 4t$

$$m^2 + 6m + 10 = 0 \Rightarrow m = \frac{-6 \pm \sqrt{36 - 4(10)}}{2}$$

$$= \frac{-6 \pm 2i}{2}$$

$$= -3 \pm i$$

$$x_c = e^{-3t}(c_1\cos t + c_2\sin t)$$

Operator that annihilates $\cos 4t$

$D^2 + 16$

$$(D^2+16)(D^2+6D+10)x=(D^2+16)(25\cos 4t)=0$$

$$x = \underbrace{e^{-3t}(c_1\cos t + c_2\sin t)}_{x_c} + \underbrace{c_3\cos 4t + c_4\sin 4t}_{x_p}$$

x_p satisfies $(D^2+6D+10)x = 25\cos 4t$

$$10\left[x_p = c_3\cos 4t + c_4\sin 4t\right]$$

$$6\left[x_p' = 4c_4\cos 4t - 4c_3\sin 4t\right]$$

$$\left[x_p'' = -16c_3\cos 4t - 16c_4\sin 4t\right]$$

$$x_p'' = -16c_3\cos 4t - 16c_4\sin 4t$$

$$6x_p' = 24c_4\cos 4t - 24c_3\sin 4t$$

$$10x_p = 10c_3\cos 4t + 10c_4\sin 4t$$

$$x_p''+6x_p'+10x_p = (-6c_3+24c_4)\cos 4t$$

$$= (-24c_3-6c_4)\sin 4t$$

$$\begin{bmatrix}-6 & 24\\ -24 & -6\end{bmatrix}\begin{bmatrix}c_3\\ c_4\end{bmatrix}=\begin{bmatrix}25\\ 0\end{bmatrix}$$

$$\begin{bmatrix}c_3\\ c_4\end{bmatrix}=\begin{bmatrix}-25/102\\ 50/51\end{bmatrix}$$

$$x_p = \frac{-25}{102}\cos 4t + \frac{50}{51}\sin 4t$$

$$x = x_c + x_p$$

$$= e^{-3t}(c_1 \cos t + c_2 \sin t) - \frac{25}{102}\cos 4t + \frac{50}{51}\sin 4t$$

$$x(0) = 1 = c_1 - \frac{25}{102} \Rightarrow c_1 = \frac{25}{102} + 1 = \frac{127}{102}$$

$$x' = e^{-3t}(-c_1 \sin t + c_2 \cos t) - 3e^{-3t}(c_1 \cos t + c_2 \sin t) + \frac{100}{102}\sin 4t + \frac{200}{51}\cos 4t$$

$$x'(0) = 0 = c_2 - 3c_1 + \frac{200}{51}$$

$$c_2 = 3c_1 - \frac{200}{51} = 3\left(\frac{127}{102}\right) - \frac{200}{51} = \frac{-19}{102}$$

Note: $x_c \to 0$ as $t \to \infty$.

When this happens, it is said to be a transient term or solution.

x_p is called the steady state solution (e.g. it doesn't go away)

$$x(t) = \underbrace{e^{-3t}\left(\frac{127}{102}\cos t - \frac{19}{102}\sin t\right)}_{x_c}$$

$$\underbrace{- \frac{25}{102}\cos 4t + \frac{50}{51}\sin 4t}_{x_p}$$