$\boxed{5.3}$ Forced Motion.

$$ma = F$$

$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt} + f(t)$$

$$\boxed{\begin{gathered}\frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{f(t)}{m}\\[1em] \frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t)\end{gathered}}$$

$$\sin^7 x$$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$
$$= \frac{1}{2} - \frac{1}{2}\cos 2x$$
$$D \qquad (D^2+4)$$

$$\sin x \cos x = \frac{1}{2}\sin 2x$$

$$\boxed{D^2+4}$$

$$\sin x \sin 2x \sin 3x$$

$$\sin(x+y) = \sin x \sin y + \sin$$

$$\sin x \sin y = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$$

Ex: Overdamped.

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 4x = 0 \qquad \underbrace{x(0)=1}_{} \qquad \underbrace{x'(0)=1}_{}$$

position of the mass is 1 unit below the equilibrium

initial velocity is 1 unit downward.

$$x(t) = \frac{5}{3}e^{-t} - \frac{2}{3}e^{-4t}$$

Find the extreme

$$x'(t) = 0$$

$$-\frac{5}{3}e^{-t} + \frac{8}{3}e^{-4t} = 0 \Rightarrow \boxed{t = \frac{1}{3}\ln\frac{8}{5} = .157}$$

$x(.157) =$ 1.069 units below the equilibrium

$$x(t) = 0$$

$$\frac{5}{3}e^{-t} - \frac{2}{3}e^{-4t} = 0$$

$$\frac{5}{3}e^{-t} = \frac{2}{3}e^{-4t}$$

$$e^{3t} = \frac{2}{5}$$

$$3t = \ln\left(\frac{2}{5}\right)$$

$t = \frac{1}{3}\ln\left(\frac{2}{5}\right) <$ negative

$\Rightarrow$ impossible

$$\frac{1}{5}\frac{d^2x}{dt^2} + 1.2\frac{dx}{dt} + 2x = 5\cos 4t$$

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 10x = 25\cos 4t$$

$$(D^2 + 6D + 10)x = 25\cos 4t$$

The roots are

$$m = \frac{-6 \pm \sqrt{36 - 4(10)}}{2} = \frac{-6 \pm 2i}{2} = -3 \pm i$$

$$x_c = c_1 e^{-3t}\cos t + c_2 e^{-3t}\sin t$$
$$= e^{-3t}(c_1 \cos t + c_2 \sin t)$$

What annihilates $25\cos 4t$

$$D^2 + 16$$

$$(D^2+16)(D^2+6D+10)x = (D^2+16)(25\cos 4t) = 0$$

$$x = \underbrace{e^{-3t}(c_1\cos t + c_2 \sin t)}_{x_c} + \underbrace{c_3\cos 4t + c_4 \sin 4t}_{x_p}$$

x_p satisfies $(D^2+6D+10)x = 25\cos 4t$.

$$10x_p = 10c_3\cos 4t + 10c_4 \sin 4t$$
$$6Dx_p = -24c_3 \sin 4t + 24c_4\cos 4t$$
$$D^2x_p = -16c_3\cos 4t - 16c_4\sin 4t$$

$$(D^2+6D+10)x_p = (-6c_3 + 24c_4)\cos 4t + (-6c_4 - 24c_3)\sin 4t$$
$$= 25\cos 4t + 0\sin 4t$$

$$-6c_3 + 24c_4 = 25$$
$$4(-24c_3 - 6c_4 = 0)$$

$$-102c_3 + 0c_4 = 25$$

$$\boxed{c_3 = \frac{-25}{102}}$$

$$\boxed{c_4 = \frac{50}{51}}$$

$$x_p = \frac{-25}{102}\cos 4t + \frac{50}{51}\sin 4t$$

$$x(t) = e^{-3t}(c_1\cos t + c_2\sin t) + -\frac{25}{102}\cos 4t + \frac{50}{51}\sin 4t$$

$$x(0) = 1 = 1(c_1 + 0c_2) + -\frac{25}{102} \Rightarrow c_1 = 1 + \frac{25}{102} = \frac{127}{102}$$

$$x'(t) = e^{-3t}(-c_1\sin t + c_2\cos t) + -3e^{-3t}(c_1\cos t + c_2\sin t) + \frac{100}{102}\sin 4t + \frac{200}{51}\cos 4t$$

$$x'(0) = c_2 - 3c_1 + \frac{200}{51} = 0$$

$$c_2 = 3c_1 - \frac{200}{51} = 3\left(\frac{127}{102}\right) - \frac{200}{51} = \frac{-19}{102}$$

$$x(t) = \underbrace{e^{-3t}\left(\frac{127}{102}\cos t - \frac{19}{102}\sin t\right)}_{x_c} \underbrace{- \frac{25}{102}\cos 4t + \frac{50}{51}\sin 4t}_{x_p}$$

Note: $x_c \to 0$ as $t \to \infty$. This is said to be the transient term

x_p is the steady state solution