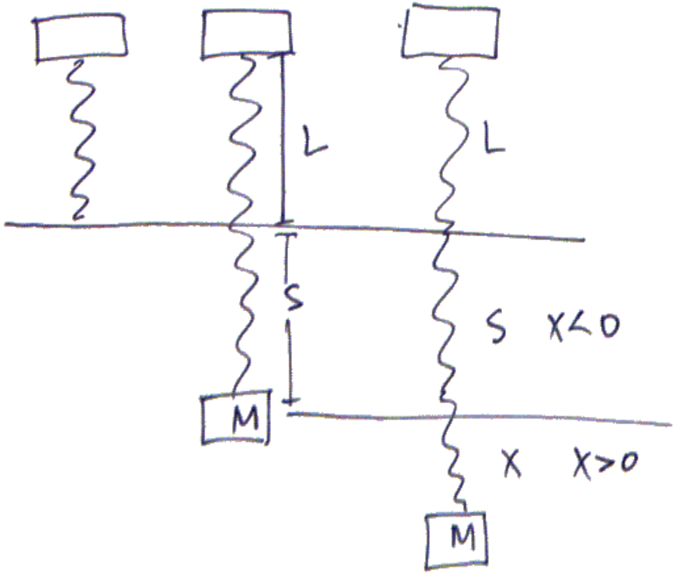


$$F = ma$$

$$m\frac{d^2x}{dt^2} = -k(s+x) + mg = -ks - kx + mg$$

$$= -kx$$

$$m\frac{d^2x}{dt^2} = -kx$$

$$\boxed{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0}$$

$$\boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0}$$

Solution: $m^2 + \omega^2 = 0 \Rightarrow m = \pm \omega i$

$$x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$$

Period: $\omega t = 2\pi \Rightarrow t = \boxed{\frac{2\pi}{\omega}}$

Frequency: $\frac{1}{\frac{2\pi}{\omega}} = \frac{\omega}{2\pi}$

Ex: Solve & interpret the IVP

$$\frac{d^2x}{dt^2} + 16x = 0 \Rightarrow x(0) = 10, \quad x'(0) = 0$$

$x(0)=10$ ↑ initial displacement

$x'(0)=0$ ↑ initial velocity.

$$x(t) = c_1 \cos 4t + c_2 \sin 4t.$$

$$x(0) = 10 = c_1 \cos 0 + \cancel{c_2 \sin 0} \Rightarrow \boxed{c_1 = 10}$$

$$x'(t) = -4c_1 \sin 4t + 4c_2 \cos 4t$$

$$x'(0) = 0 = \cancel{-4c_1 \sin 0} + 4c_2 \cos 0 \Rightarrow \boxed{c_2 = 0}$$

$$\boxed{x(t) = 10 \cos 4t}$$

Alternative form of $x(t)$

$$c_1 \sin \omega t + c_2 \cos \omega t$$

$$\sin(x+y) = \sin x \cos y + \sin y \cos x.$$

$$A \sin(\omega t + \phi) = A[\sin \omega t \cos \phi + \sin \phi \cos \omega t]$$

$$= \underbrace{A\cos\phi}_{c_1} \sin \omega t + \underbrace{A \sin \phi}_{c_2} \cos(\omega t)$$

$\boxed{5.2}$ Damped Motion

$$\frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

or $$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

$$m^2 + 2\lambda m + \omega^2 = 0$$

$$m = \frac{-2\lambda \pm \sqrt{4\lambda^2 - 4\omega^2}}{2} = -\lambda \pm \sqrt{\lambda^2-\omega^2}$$

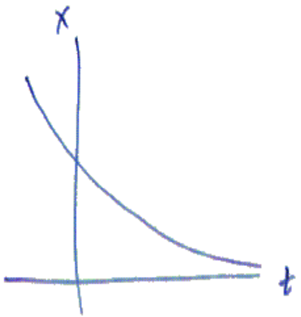
$$m_1 = -\lambda + \sqrt{\lambda^2-\omega^2} \qquad m_2 = -\lambda - \sqrt{\lambda^2-\omega^2}$$

$\boxed{\text{Case I}}$ $\lambda^2 - \omega^2 > 0$ (2 real roots)

$$x(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t}$$

$$= c_1 e^{(-\lambda+\sqrt{\lambda^2-\omega^2})t} + c_2 e^{(-\lambda-\sqrt{\lambda^2-\omega^2})t}$$

$$= e^{-\lambda t}\left[c_1 e^{\sqrt{\lambda^2-\omega^2}\,t} + c_2 e^{-\sqrt{\lambda^2-\omega^2}\,t}\right]$$



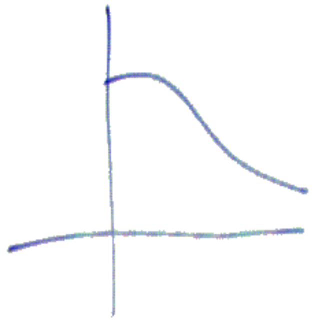
$\boxed{\text{Case II}}$ $\lambda^2-\omega^2=0$ (Repeate Real Roots)

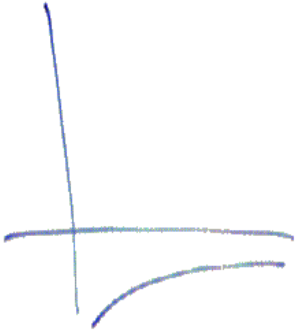
$$x(t)=c_1e^{m_1t}+c_2te^{m_1t}$$

System is critically damped

Any slight decrease in the damping force would result in oscillatory motion.

$$x(t)=(c_1+c_2t)e^{m_1t}$$





$\boxed{\text{Case III}}$ $\lambda^2-\omega^2<0$ $\Rightarrow$ complex roots

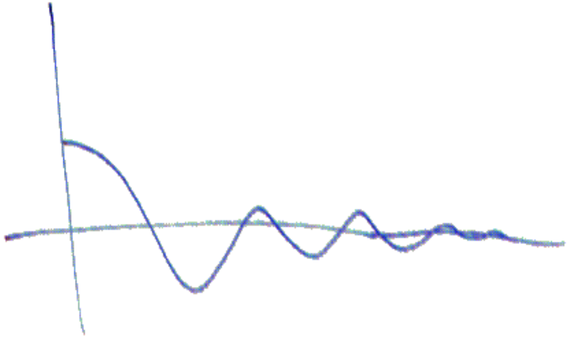
underdamped

$$x(t)=c_1e^{\alpha t}\cos\beta t+c_2e^{\alpha t}\sin\beta t$$

$$m=-\lambda\pm\sqrt{\lambda^2-\omega^2}$$

$$=\underbrace{-\lambda}_{\alpha}\pm i\underbrace{\sqrt{\omega^2-\lambda^2}}_{\beta}$$

$$x(t)=c_1e^{-\lambda t}\cos\left[\sqrt{\omega^2-\lambda^2}\,t\right]+c_2e^{-\lambda t}\sin[\text{same thing}]$$



Ex: Overdamped

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 4x = 0$$

$$\underbrace{x(0)=1}_{\text{position of the mass is 1 unit below the equilibrium}} \qquad \underbrace{x'(0)=1}_{\text{initial velocity is 1 unit downward.}}$$

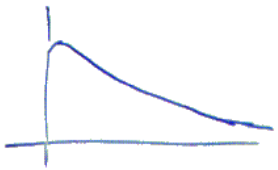
$$x(t) = \frac{5}{3}e^{-t} - \frac{2}{3}e^{-4t}$$

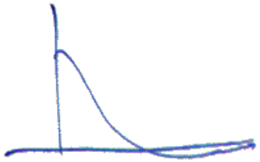
Find the extreme

$$x'(t) = 0$$

$$-\frac{5}{3}e^{-t} + \frac{8}{3}e^{-4t} = 0 \Rightarrow \boxed{t = \frac{1}{3}\ln\frac{8}{5} = .157}$$

$x(.157) =$ 1.069 units below the equilibrium





$$x(t) = 0$$

$$\frac{5}{3}e^{-t} - \frac{2}{3}e^{-4t} = 0$$

$$\frac{5}{3}e^{-t} = \frac{2}{3}e^{-4t}$$

$$e^{3t} = \frac{2}{5}$$

$$3t = \ln\left(\frac{2}{5}\right)$$

$$t = \frac{1}{3}\ln\left(\frac{2}{5}\right) < \text{negative}$$

$\Rightarrow$ impossible.