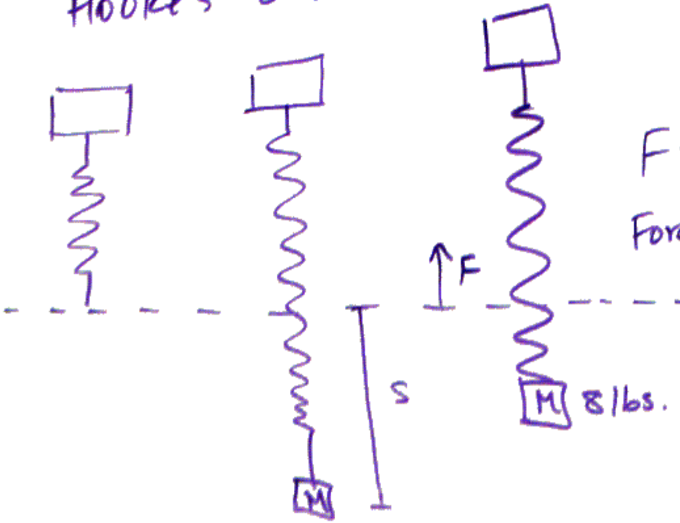
5.1 Simple Harmonic Motion

Hooke's Law



$F = ks$

Force is proportial to the distance stretched.

resting spot

k = prop. constant

s = elogation amount

F = acts opposite to s.

A pull in the downward direction is positive.

Newton's Second Law

Equilibrium is established.

Weight matches upward force

$W = mg$ $\quad$ $F = ks$

$mg = ks$

or $\quad mg - ks = 0$

If it is pulled and released, then Newton's second law applies

$F = ma$

So

$$m\frac{d^2s}{dt^2} = -k(s+x) + mg$$

$$= -kx + \underbrace{mg - ks}_{0} = -kx$$

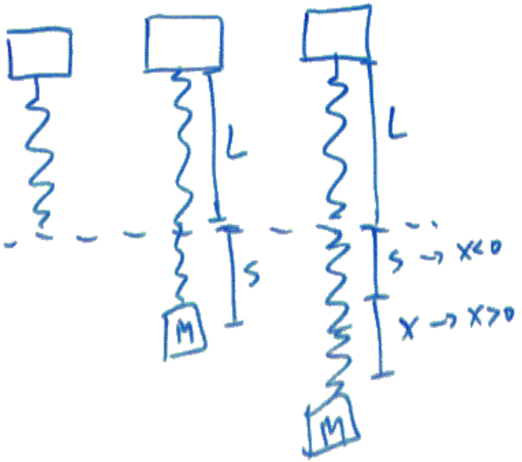
So

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

or

$$\boxed{\frac{d^2x}{dt^2} + \frac{k}{m}x = 0}$$

$$\frac{d^2x}{dt^2} + w^2 x = 0$$



This equation describes simple harmonic motion, or free undamped motion.

Initial conditions:

Initial displacement $x(0) = \alpha$

Initial velocity $x'(0) = \beta$

Solution:

$$x(t) = c_1\cos(wt) + c_2\sin(wt)$$

Period → $T = \frac{2\pi}{w}$

Frequency → $f = \frac{1}{T} = \frac{w}{2\pi}$

Ex: Solve & Interpret the IVP

$$\frac{d^2x}{dt^2} + 16x = 0 \qquad x(0) = 10, \quad x'(0) = 0$$

$$x(t) = c_1 \cos 4t + c_2 \sin 4t$$

$$x(0) = c_1 = 10 \Rightarrow \boxed{c_1 = 10}$$

$$x'(t) = -4c_1 \sin 4t + 4c_2 \cos 4t$$

$$x'(0) = 0 = 4c_2 \cos(0) \Rightarrow c_2 = 0$$

$$x(t) = 10 \cos 4t$$

Alternative form of $x(t)$

$$x(t) = c_1 \sin \omega t + c_2 \cos \omega t$$

Note: $\sin(x+y) = \sin x \cos y + \sin y \cos x$

and

$$A \sin(\omega t + \phi) = A[\sin(\omega t)\cos\phi + \sin\phi \cos \omega t]$$

$$= \underbrace{A\cos\phi}_{c_1} \sin \omega t + \underbrace{A\sin\phi}_{c_2} \cos \omega t$$

$c_1 = A\cos\phi \qquad c_2 = A\sin\phi$

$\frac{c_1}{A} = \cos\phi \qquad \frac{c_2}{A} = \sin\phi$

$$A = \sqrt{c_1^2 + c_2^2} \qquad \tan\phi = \frac{c_2}{c_1}$$

So

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$$= -kx + \underbrace{mg-ks}_{0} = -kx$$

So

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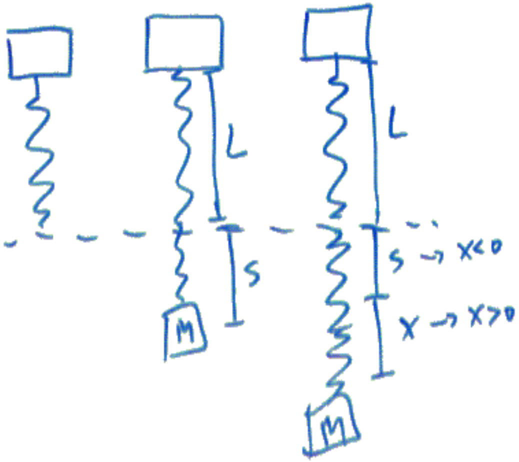

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$\boxed{5.2}$ Damped Motion

$$\frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

$$m^2 + 2\lambda m + \omega^2 = 0$$

$$m = \frac{-2\lambda \pm \sqrt{4\lambda^2 - 4\omega^2}}{2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2}$$

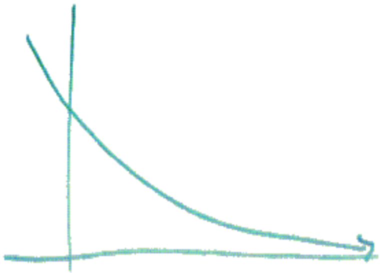
Three cases

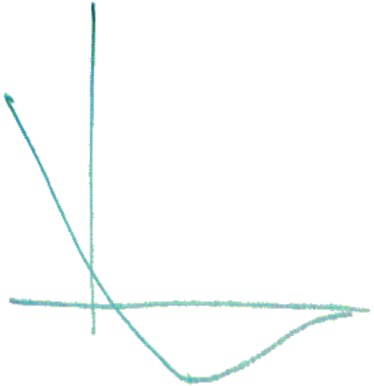
Case I $\quad \lambda^2 - \omega^2 > 0 \Rightarrow$ Two real roots.

$$x(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t}$$

$$= c_1 e^{(-\lambda + \sqrt{\lambda^2 - \omega^2})t} + c_2 e^{(-\lambda - \sqrt{\lambda^2 - \omega^2})t}$$

$$x(t) = e^{-\lambda t}\left[c_1 e^{\sqrt{\lambda^2 - \omega^2}\,t} + c_2 e^{-\sqrt{\lambda^2 - \omega^2}\,t}\right]$$





Case II $\lambda^2=\omega^2$ (One repeated root)

$$x(t) = c_1 e^{m_1 t} + c_2 t e^{m_1 t}$$

System is $\boxed{\text{"critically damped"}}$

(any slight decrease in the damping force would result in oscillatory motion).

$$\boxed{x(t) = (c_1 + c_2 t)e^{-\lambda t}}$$

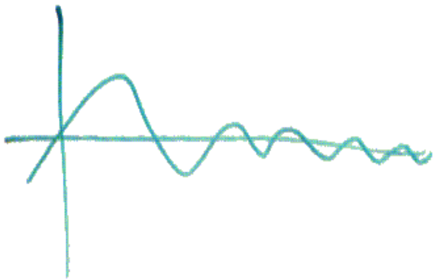
Case III $\lambda^2-\omega^2<0 \Rightarrow$ complex roots. $\boxed{\text{Underdamped}}$

$$m = -\lambda \pm \sqrt{\lambda^2-\omega^2}$$

$$= \underbrace{-\lambda}_{\alpha} \pm i \underbrace{\sqrt{\omega^2-\lambda^2}}_{\beta}$$

$$x(t) = c_1 e^{-\lambda t}\cos\left[\sqrt{\omega^2-\lambda^2}\, t\right] + c_2 e^{-\lambda t}\sin\left[\sqrt{\omega^2-\lambda^2}\, t\right]$$

$$= e^{-\lambda t}\left[c_1\cos\left[\sqrt{\omega^2-\lambda^2}\, t\right] + c_2\sin\left[\sqrt{\omega^2-\lambda^2}\, t\right]\right]$$



Ex: Overdamped.

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 4x = 0$$

$x(0) = 1 \to$ 1 position unit below equilibrium

$x'(0) = 1 \to$ downward velocity of 1.

Find the extremum

$$x(t) = \frac{5}{3}e^{-t} - \frac{2}{3}e^{-4t}$$

$$x'(t) = -\frac{5}{3}e^{-t} + \frac{8}{3}e^{-4t} = 0$$

$$t = \frac{1}{3}\ln\left(\frac{8}{5}\right) = .157$$

Obtains an extreme displacement of $x(.157) = 1.069$ units.

Does it pass the equilibrium pt

$$x(t) = 0$$

$$\frac{5}{3}e^{-t} - \frac{2}{3}e^{-4t} = 0$$

$$\frac{5}{3}e^{-t} = \frac{2}{3}e^{-4t}$$

$$e^{3t} = \frac{2}{5}$$

$$3t = \ln\left(\frac{2}{3}\right)$$

$$t = \frac{1}{3}\ln\left(\frac{2}{5}\right) = -.305$$

impossible!

$\boxed{5.3}$ Forced Motion.

$$ma = F$$

$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt} + f(t)$$

or $$\frac{d^2x}{dt^2} + \frac{\beta}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{f(t)}{m}$$

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t)$$

Can solve using Sec 4.6 or 4.7.

Ex: Interpret & solve.

$$\frac{1}{5}\frac{d^2x}{dt^2} + 1.2\frac{dx}{dt} + 2x = 5\cos 4t$$

$\underbrace{x(0)=1}_{\text{1 unit below equilibrium.}}$ $\underbrace{x'(0)=0}_{\text{starts at rest}}$

$\frac{1}{5}$ = mass (slug or kg)

$k=2 \Rightarrow$ spring const. (lb/ft N/m)

$\beta=1.2 \Rightarrow$ Motion is damped.

$5\cos 4t$ = external force with a period of $\frac{\pi}{2}$.

Solve:

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 10x = 25\cos 4t$$

$$(D^2+6D+10)x = 25\cos 4t \quad (*)$$

$$m^2+6m+10=0 \Rightarrow m=-3\pm i$$

$$x_c = e^{-3t}(c_1\cos t + c_2\sin t)$$

Operator that annihilates $\cos 4t$ $\quad D^2+16$

$$(D^2+16)(D^2+6D+10)x = 0$$

$$x = \underbrace{e^{-3t}(c_1\cos t + c_2\sin t)}_{x_c} + \underbrace{c_3\cos 4t + c_4\sin 4t}_{x_p}$$

x_p satisfies $(*)$

$$10x_p = [c_3\cos 4t + c_4\sin 4t]\,10$$

$$6x_p' = [4c_4\cos 4t - 4c_3\sin 4t]\,6$$

$$x_p'' = -16c_3\cos 4t - 16c_4\sin 4t$$

$$x_p'' + 6x_p' + 10x_p$$

$$= [-16c_3 + 24c_4 + 10c_3]\cos 4t$$

$$+ [10c_4 - 24c_3 - 16c_4]\sin 4t$$

$$= [-6c_3 + 24c_4]\cos 4t$$

$$+ [-24c_3 - 6c_4]\sin 4t$$

$$\Rightarrow \begin{aligned} -6c_3+24c_4 &= 25 \\ -24c_3-6c_4 &= 0 \end{aligned}$$

$$\boxed{c_3 = \frac{-25}{102}}$$

$$\boxed{c_4 = \frac{50}{51}}$$