4.7 Variation of Parameters

$$\sum_{k=0}^{n} a_k y^{(k)} = g(x) \rightarrow g(x) \text{ was restricted in 4.6.}$$

Motivation:

The solution of $\frac{dy}{dx} + P(x)y = f(x)$ is

$$y = \underbrace{e^{-\int P(x)dx} \int e^{\int P(x)dx} f(x)\,dx}_{y_p} + \underbrace{c_1 e^{-\int P(x)dx}}_{y_c}$$

y_c is a solution to $\frac{dy}{dx} + P(x)y = 0$

Variation of Parameters

Suppose that y_1 is a known solution of the homogeneous system

$$\frac{dy}{dx} + P(x)y = 0$$

Hence, $y_1 = e^{-\int P(x)dx}$

Variation of Parameters consists of finding u_1 such that

$$y_p = u_1 y_1(x)$$

is a particular solution of

$$\frac{dy}{dx} + P(x)y = f(x)$$

So: Substitute $y_p = u_1 y_1(x)$ into

$$\frac{dy}{dx} + P(x)y = f(x)$$

$$f(x) = \frac{d(u_1 y_1)}{dx} + P(x) u_1 y_1$$

$$= u_1 \frac{dy_1}{dx} + y_1 \frac{du_1}{dx} + P(x) u_1 y_1$$

$$= u_1 \underbrace{\left[\frac{dy_1}{dx} + P(x) y_1 \right]}_{=0} + y_1 \frac{du_1}{dx}$$

$$y_1 \frac{du_1}{dx} = f(x)$$

$$du_1 = \frac{f(x)\,dx}{y_1}$$

$$u_1 = \int \frac{f(x)\,dx}{y_1}$$

So $y_p = u_1 y_1 = y_1 \int \frac{f(x)\,dx}{y_1} = e^{-\int P(x)dx} \int \frac{f(x)\,dx}{e^{-\int P(x)dx}}$

Adapt this method to second order equations

$$a_2(x) y'' + a_1(x) y' + a_0(x) y = g(x)$$

$$y'' + P(x) y' + Q(x) y = f(x)$$

$$y_p = u_1 y_1 + u_2 y_2 = u_1(x) y_1(x) + u_2(x) y_2(x)$$

y_1 & y_2 are sol. to homogeneous system.

Because we need to find u_1 & u_2, then we need at least two equations.

the choice of $\boxed{y_1u_1' + y_2u_2' = 0}$ will simplify the derivative and the second derivative of y_p

$$y_p = u_1y_1 + u_2y_2$$

$$y_p' = u_1y_1' + y_1u_1' + u_2y_2' + y_2u_2'$$

$$= u_1y_1' + u_2y_2' + \underbrace{y_1u_1' + y_2u_2'}_{=0}$$

$$\boxed{y_p' = u_1y_1' + u_2y_2'}$$

$$y_p'' = u_1y_1'' + u_1'y_1' + y_2'u_2' + u_2y_2''$$

$$f(x) = y_p'' + P(x)y_p' + Q(x)y_p$$

$$= \boxed{u_1y_1''} + u_1'y_1' + y_2'u_2' + \boxed{u_2y_2''}$$

$$+ P(x)\left[\boxed{u_1y_1'} + \boxed{u_2y_2'}\right]$$

$$+ Q(x)\left[\boxed{u_1y_1} + \boxed{u_2y_2}\right]$$

$$= u_1\left[\underbrace{y_1'' + P(x)y_1' + Q(x)y_1}_{0}\right]$$

$$+ u_2\left[\underbrace{y_2'' + P(x)y_2' + Q(x)y_2}_{0}\right]$$

$$+ u_1'y_1' + y_2'u_2'$$

$$\boxed{f(x) = u_1'y_1' + u_2'y_2'}$$

So we have:

$$y_1 u_1' + y_2 u_2' = 0 \quad \text{and}$$

$$y_1' u_1' + y_2' u_2' = f(x).$$

The system can be solved using Cramer's Rule:

where $u_1' = \dfrac{W_1}{W}$, $u_2' = \dfrac{W_2}{W}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 \\ f & y_2' \end{vmatrix} \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f \end{vmatrix}$$

In General:

$$y^{(n)} + P_{n-1}(x) y^{(n-1)} + \dots + P_1(x) y' + P_0(x) y = f(x)$$

$$y_c = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$$

is the sol. to the homogeneous system.

$$y_p = u_1 y_1 + u_2 y_2 + \dots + u_n y_n$$

$$\begin{aligned}
y_1 u_1' + y_2 u_2' + \dots + y_n u_n' &= 0 \\
y_1' u_1' + y_2' u_2' + \dots + y_n' u_n' &= 0 \\
&\vdots \\
y_1^{(n-2)} u_1' + y_2^{(n-2)} u_2' + \dots + y_n^{(n-2)} u_n' &= 0 \\
y_1^{(n-1)} u_1' + y_2^{(n-1)} u_2' + \dots + y_n^{(n-1)} u_n' &= f(x)
\end{aligned}$$

$$u_k' = \frac{W_k}{W}, \text{ where } W = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & & y_n' \\ \vdots & & \ddots & \vdots \\ y_1^{(n-1)} & \cdots & & y_n^{(n-1)} \end{vmatrix}$$

W_k is W with k^{th} col replaced by $\begin{bmatrix} 0 \\ \vdots \\ f \end{bmatrix}$

Integrate u_k' and

$$y = y_c + y_p$$

Ex: Solve $y'' - 2y' + 2y = e^x \sec x$

Since $e^x \sec x$ cannot be annihilated, then we must use variation of parameters

Solve homogeneous:

$$m^2 - 2m + 2 = 0 \Rightarrow m = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

$$y_c = c_1 e^x \cos x + c_2 e^x \sin x$$

$y_1 = e^x \cos x$ $\quad$ $y_2 = e^x \sin x$

$y_1' = e^x(\cos x - \sin x)$ $\quad$ $y_2' = e^x(\sin x + \cos x)$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x(\cos x - \sin x) & e^x(\sin x + \cos x) \end{vmatrix}$$

$$= e^{2x}(\sin x \cos x + \cos^2 x) - e^{2x}(\sin x \cos x - \sin^2 x)$$

$$= e^{2x}$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f & y_2' \end{vmatrix} = \begin{vmatrix} 0 & e^x \sin x \\ e^x \sec x & e^x(\sin x + \cos x) \end{vmatrix}$$

$$= -e^{2x} \tan x$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f \end{vmatrix} = \begin{vmatrix} e^x \cos x & 0 \\ e^x(\cos x - \sin x) & e^x \sec x \end{vmatrix} = e^{2x}$$

$$\boxed{W_1 = -e^{2x} \tan x} \quad \boxed{W_2 = e^{2x}}$$

$$u_1' = \frac{W_1}{W} = \frac{-e^{2x} \tan x}{e^{2x}} = -\tan x$$

$$u_2' = \frac{W_2}{W} = \frac{e^{2x}}{e^{2x}} = 1$$

$$u_1 = \int -\tan x \, dx = \int -\frac{\sin x}{\cos x} \, dx = \ln|\cos x| + C_3$$

$$u_2 = \int dx = x + C_4$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= (\ln|\cos x| + C_3) e^x \cos x + (x + C_4) e^x \sin x$$

$$= \underbrace{\ln|\cos x| e^x \cos x + x e^x \sin x}_{y_p} + \underbrace{C_3 e^x \cos x + C_4 e^x \sin x}_{\text{is this needed?}}$$

No! Because this is y_c.

General solution:

$$y = y_p + y_c$$

Note: when finding u_1 & u_2 you don't need the constants.

Ex: $y'' + 2y' + y = e^{-x} \ln x$

Solution to homogeneous:

$m^2 + 2m + 1 = (m+1)^2 \Rightarrow m = -1$

$y_c = c_1 e^{-x} + c_2 x e^{-x}$

$y_p = u_1 e^{-x} + u_2 x e^{-x}$

Solve: $u_1' = \dfrac{W_1}{W}$ and $u_2' = \dfrac{W_2}{W}$

$$W = \begin{vmatrix} e^{-x} & xe^{-x} \\ -e^{-x} & -xe^{-x} + e^{-x} \end{vmatrix} = \cancel{-xe^{-2x}} + e^{-2x} + \cancel{xe^{-2x}} = e^{-2x}$$

$$W_1 = \begin{vmatrix} 0 & xe^{-x} \\ e^{-x}\ln x & \sim \end{vmatrix} = -xe^{-2x} \ln x$$

$$W_2 = \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & e^{-x}\ln x \end{vmatrix} = e^{-2x} \ln x$$

$$u_1' = \frac{W_1}{W} = \frac{-xe^{-2x}\ln x}{e^{-2x}} = -x \ln x$$

$$u_2' = \frac{W_2}{W} = \frac{e^{-2x}\ln x}{e^{-2x}} = \ln x$$

$u_1 = \int -x \ln x \, dx$

$u_2 = \int \ln x \, dx$

$$u_1 = \int -x \ln x \, dx = -\frac{x^2}{2}\ln x + \int \frac{x}{2}\, dx$$

$u = \ln x \qquad dv = -x\,dx$

$du = \frac{1}{x}dx \qquad v = -\frac{x^2}{2}$

$$= -\frac{x^2}{2}\ln x + \frac{x^2}{4}$$

$$u_2 = \int \ln x \, dx = x\ln x - \int dx = x\ln x - x$$

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x}dx \qquad v = x$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \left(-\frac{x^2}{2}\ln x + \frac{x^2}{4}\right)e^{-x} + (x\ln x - x)\,x e^{-x}$$

$$= -\frac{1}{2}x^2 \ln x\, e^{-x} + \frac{1}{4}x^2 e^{-x} + x^2 \ln x\, e^{-x} - x^2 e^{-x}$$

$$= \frac{1}{2}x^2 \ln x\, e^{-x} - \frac{3}{4}x^2 e^{-x}$$

$$y = y_p + y_c$$

$$= \frac{1}{2}x^2 \ln x\, e^{-x} - \frac{3}{4}x^2 e^{-x} + c_1 e^{-x} + c_2 x e^{-x}$$