$\boxed{4.6}$ Undetermined Coefficients (Annihilator approach)

Note: the n^{th} order linear diff-eq

$$(*) \quad \sum_{k=0}^{n} a_k y^{(n)} = g(x) \qquad \text{can be written as}$$

$$Ly = g(x) \qquad L \text{ is the linear}$$

differential operator that corresponds the left side of $*$.

The method that is presented covers these functions for $g(x)$:

a) coefficients are constant (a_k's)

b) $g(x)$ is
1) constant
2) polynomial
3) exponential
4) $\sin \alpha x$
5) $\cos \alpha x$
6) finite sums & products of these.

functions that can be annihilated:

$g(x)=10$; $g(x)=x^2+10x$

$g(x)=e^x\cos x$

$g(x)=e^x+x^2e^x\cos x+(3x^2+10)\sin x$

But it won't work for:

$g(x)=\ln x$; $g(x)=\frac{1}{x}$, $g(x)=\tan x$

$g(x)=\sin^{-1}x$

Ex: $2\frac{d^2y}{dx^2}+14\frac{dy}{dx}+20y=5x^3$ (1)

Solve the homogeneous system!

$$2y''+14y'+20y=0$$

$$(2D^2+14D+20)y=0$$

$$2(D^2+7D+10)y=0$$

$$2(D+2)(D+5)y=0$$

y_c = complentary function (solution the homogeneous)

$$\boxed{y_c=c_1e^{-2x}+c_2e^{-5x}}$$

What annihilates $5x^3$?

Comb (1) & (2)

$$2(D+5)(D+2)y = 5x^3$$

$$2D^4(D+5)(D+2)y = D^4(5x^3) = 0$$

$$y = \underbrace{c_3 + c_4 x + c_5 x^2 + c_6 x^3}_{y_p} + \underbrace{c_1 e^{-2x} + c_2 e^{-5x}}_{y_c}$$

y_p satisfies (1)

y_p should satisfy $(2D^2+14D+20)y = 5x^3$

(since it is a solution)

$$y_p = A + Bx + Cx^2 + Ex^3$$

$$y_p' = B + 2Cx + 3Ex^2$$

$$y_p'' = 2C + 6Ex$$

$$5x^3 = (2D^2 + 14D + 20)y_p$$

$$= 2D^2 y_p + 14 D y_p + 20 y_p$$

$$= 2y_p'' + 14 y_p' + 20 y_p$$

$$= 2(2C + 6Ex) + 14(B + 2Cx + 3Ex^2) + 20(A + Bx + Cx^2 + Ex^3)$$

Combine according to powers of x.

$$= (4C + 14B + 20A) + (12E + 28C + 20B)x + (42E + 20C)x^2 + \underline{20E}\,x^3$$

Solve:

$$4C + 14B + 20A = 0$$

$$12E + 28C + 20B = 0$$

$$42E + 20C = 0$$

$$20E = 5$$

$$\begin{bmatrix} 20 & 14 & 4 & 0 \\ 0 & 20 & 28 & 12 \\ 0 & 0 & 20 & 42 \\ 0 & 0 & 0 & 20 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \\ C \\ E \end{bmatrix} = \begin{bmatrix} 20 & 14 & 4 & \\ 0 & 20 & 28 & 12 \\ 0 & 0 & 20 & 42 \\ 0 & 0 & 0 & 20 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \end{bmatrix}$$

$$= \left(\frac{-609}{2000}, \frac{117}{200}, \frac{-21}{40}, \frac{1}{4} \right)$$

Thus, $y_p = \frac{-609}{2000} + \frac{117}{200}x - \frac{21}{40}x^2 + \frac{1}{4}x^3$

And the solution to (1) is

$$y = y_p + y_c$$
$$= \frac{-609}{2000} + \frac{117}{200}x - \frac{21}{40}x^2 + \frac{1}{4}x^3 + c_1 e^{-2x} + c_2 e^{-5x}$$

The diff-eq $L(y)=g(x)$

L has constant coeff.

$g(x)$ has finite sums & products of constants, polynomials, $e^{\alpha x}$, sines & cosines.

(i) Find y_c (solution to $Ly=0$)

(ii) Find the operator L_1 which annihilates $g(x)$

Operate both sides of $Ly=g(x)$

$$\underbrace{L_1 L}\, y = L_1 g(x) = 0$$

(iii) Solve $L_1 Ly=0$

(iv) Determine in ↑ parts that are duplicated in y_c

Form of y_p is known

(v) Substitute y_p into $Ly=g(x)$. Match coeff & solve

(vi) $y=y_p+y_c$

Ex: $y'' + y = 4\cos x + 3\sin x - 8$

Homogeneous

$y'' + y = 0 \Rightarrow (D^2+1)y = 0$

$m = \pm i$

$y_c = c_2 \cos x + c_3 \sin x$

(D^2+1) annihilates $\sin x$ & $\cos x$

$D \qquad L_1 = D(D^2+1)$

$(D^2+1)y = 4\cos x + 3\sin x - 8$

$D(D^2+1)(D^2+1)y = D(D^2+1)(4\cos x + 3\sin x - 8)$

$= 0$

$D(D^2+1)^2 y = 0$

$$y = \underbrace{C_1 + \overbrace{\boxed{C_2\cos x + C_3 \sin x}}^{y_c} + C_4 x\cos x + C_5 x \sin x}_{y_p}$$

$y_p = C_1 + C_4 x\cos x + C_5 x\sin x$

$y_p' = C_4(-x\sin x + \cos x)$
$\quad + C_5(x\cos x + \sin x)$

$y_p'' = C_4[-x\cos x - \sin x - \sin x]$
$\quad + C_5[-x\sin x + \cos x + \cos x]$

$$(D^2+1)y_p = 4\cos x + 3\sin x - 8$$

$$D^2_{y_p} + y_p = c_4[-\cancel{x\cos x} - 2\sin x] + c_5[\cancel{-x\sin x} + 2\cos x]$$

$$+ c_1 + \cancel{c_4 x\cos x} + \cancel{c_5 x\sin x}.$$

$$= c_1 - 2c_4\sin x + 2c_5\cos x = 4\cos x + 3\sin x - 8$$

$$c_1 = -8$$

$$-2c_4 = 3 \Rightarrow c_4 = -3/2$$

$$2c_5 = 4 \Rightarrow c_5 = 2$$

$$y_p = -8 - \frac{3}{2}x\cos x + 2x\sin x$$

$$y_c = C_2\cos x + C_3\sin x$$

Solution is:

$$y = y_p + y_c$$

$$= -8 - \frac{3}{2}x\cos x + 2x\sin x$$

$$+ C_2\cos x + C_3\sin x.$$