$\boxed{4.6}$ Undetermined Coefficients (Annihilator approach)

Note that the $n^{th}$ order linear diff-eq.

$$(*) \quad \sum_{k=0}^{n} a_k y^{(k)} = g(x) \quad \text{can}$$

also be written as $Ly = g(x)$.

$L$ is the linear diff operator.

This method we will talk about is limited to

a) coefficients that are constant

b) $g(x)$ is only these kinds of functions:

1) constants
2) polynomials
3) exponential functions
4) $\sin \alpha x$ & $\cos \alpha x$
5) finite sums of these.

Note: this approach will work if $g(x) = 10$, $g(x) = x^2 + 10x$, $g(x) = e^x$, $g(x) = e^x \cos x$, $g(x) = e^x \sin x + x^2 e^x \cos x$

But won't work for $g(x) = \ln x$, $g(x) = \frac{1}{x}$, $g(x) = \tan x$, etc.

The diff-eq $Ly = g(x)$

(constant coefficients)

$g(x)$ has finite sums & products of constants, polynomials, $e^{\alpha x}$, sines & cosines.

(i) Find $y_c$ (Solution to $Ly=0$)

(ii) Find the operator which annihilates $g(x)$ (call it $L_1$). Operate both sides of $Ly=g(x)$ by $L_1$ $\Rightarrow$ $L_1 L y = L_1 g(x) = 0$

(iii) Solve $L_1 L y = 0$

(iv) Delete terms in ↑ that are duplicated in $y_c$.

Form of $y_p$ is known.

(v) Substitute $y_p$ into $Ly=g(x)$

Match coeffs & solve.

(vi) $y = y_c + y_p$.

4.6 cont.

The diff-eq $Ly = g(x)$ with $L$ containing constant coefficients. $g(x)$ has finite sums & products of constants, polynomials, $e^{ax}$, sines, and cosines.

(i) Find $y_c$ (solution to $Ly=0$)

(ii) Find the operator which annihilates $g(x)$ (call it $L_1$) Operate both sides of $Ly=g(x)$ by $L_1$

$$L_1 L y = L_1 g(x) = 0$$

(iii) solve $L_1 L y = 0$

(iv) The previous solution determines terms that are duplicated in $y_c$ what is left is $y_p$.

(v) Substitute $y_p$ into $Ly=g(x)$. Match coefficients & solve

(vi) The solution to $Ly=g(x)$ is $y = y_c + y_p$

Solve $2y'' + 14y' + 20y = 5x^3$ (1)

Let $L = 2D'' + 14D + 20$

① Solve $Ly = 0$.

$$2m^2 + 14m + 20 = 0$$
$$m^2 + 7m + 10 = 0$$
$$(m+5)(m+2) = 0$$
$$\Rightarrow m = -5, -2.$$

$$y_c = c_1 e^{-2x} + c_2 e^{-5x}$$

② what annihilates $5x^3 \Rightarrow D^4$.

Thus, we have

$$Ly = g(x)$$
$$D^4(2D^2 + 14D + 20)y = D^4 g(x) = D^4(5x^3) = 0$$

Solve $D^4(2D^2 + 14D + 20)y = 0$.

$$D^4(D^2 + 7D + 10)y = 0$$
$$D^4(D+5)(D+2)y = 0$$

③ $$y = \underbrace{c_1 e^{-2x} + c_2 e^{-5x}}_{y_c} + \underbrace{c_3 + c_4 x + c_5 x^2 + c_6 x^3}_{y_p}$$

④

⑤ $$5x^3 = (2D'' + 14D + 20)y_p$$
$$= (2D'' + 14D + 20)(c_3 + c_4 x + c_5 x^2 + c_6 x^3)$$
$$= 2D''(c_3 + c_4 x + c_5 x^2 + c_6 x^3)$$
$$+ 14D(c_3 + c_4 x + c_5 x^2 + c_6 x^3)$$
$$+ 20(c_3 + c_4 x + c_5 x^2 + c_6 x^3)$$

$$= \begin{array}{l} 2(0 + 0 + \cancel{2c_5} + \cancel{6c_6 x}) \\ + 14(0 + \cancel{c_4} + \cancel{2c_5 x} + 3c_6 x^2) \\ + 20(\cancel{c_3} + \cancel{c_4 x} + c_5 x^2 + c_6 x^3) \end{array} = \begin{array}{l} (4c_5 + 14c_4 + 20c_3) \\ (12c_6 + 28c_5 + 20c_4)x \\ (42c_6 + 20c_5)x^2 + 20c_6 x^3 \end{array}$$

$$5x^3 = (4c_5 + 14c_4 + 20c_3) + (12c_6 + 28c_5 + 20c_4)x$$

$$+ (42c_6 + 20c_5)x^2 + 20c_6 x^3$$

$$5 = 20c_6 \Rightarrow \boxed{c_6 = \frac{1}{4}}$$

$$0 = 42c_6 + 20c_5 \Rightarrow 0 = \frac{42}{4} + 20c_5 \Rightarrow -\frac{21}{2} = 20c_5 \Rightarrow \boxed{c_5 = -\frac{21}{40}}$$

$$0 = 12c_6 + 28c_5 + 20c_4 \Rightarrow 0 = 12\left(\frac{1}{4}\right) + 28\left(\frac{-21}{40}\right) + 20c_4 \Rightarrow \boxed{c_4 = \frac{-12/4 + \frac{28(21)}{40}}{20} = \frac{117}{200}}$$

$$0 = 4c_5 + 14c_4 + 20c_3 \Rightarrow \boxed{A = \frac{-609}{2000}}$$

Thus, $y_p = -\frac{609}{2000} + \frac{117}{200}x - \frac{21}{40}x^2 + \frac{1}{4}x^3$, $y_c = c_1 e^{-2x} + c_2 e^{-5x}$

and the general solution is

$$y = y_c + y_p$$

$$y'' + 4y = 4\cos x + 3\sin x - 8$$

$$L = D^2 + 4$$

① Solve $(D^2+4)y = 0$

$$m^2 + 4 = 0 \Rightarrow m = \pm 2i \quad \alpha = 0,\ \beta = 2$$

$$y_c = c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x$$

$$= c_1\cos 2x + c_2\sin 2x$$

② What annihilates $g(x) = 4\cos x + 3\sin x - 8$

$$\underbrace{D^2+1 \quad D^2+1 \quad D}$$

$$L_1 = D(D^2+1)$$

Apply $L_1$ to $Ly = g(x)$.

$$(D^2+4)y = 4\cos x + 3\sin x - 8$$

$$D(D^2+1)(D^2+4)y = D(D^2+1)(4\cos x + 3\sin x - 8) = 0$$

$$m = 0, \pm i, \pm 2i$$

$$y = \underbrace{c_1\cos 2x + c_2\sin 2x}_{y_c} + \underbrace{c_3 + c_4\cos x + c_5\sin x}_{y_p}$$

Put $y_p$ into $Ly = g(x)$.

$$\overbrace{4\cos x + 3\sin x - 8}^{g(x)} = (D^2+4)y_p = (D^2+4)(c_3 + c_4\cos x + c_5\sin x)$$

$$= -c_4\cos x - c_5\sin x + 4c_3 + 4c_4\cos x + 4c_5\sin x$$

$$= 3c_4\cos x + 3c_5\sin x + 4c_3$$

$$4c_3 = -8 \qquad 3 = 3c_5 \qquad 4 = 3c_4$$

$$\boxed{c_3 = -2} \qquad \boxed{c_5 = 1} \qquad \boxed{c_4 = \frac{4}{3}}$$

So the general solution is

$$\boxed{y = c_1\cos 2x + c_2\sin 2x + \frac{4}{3}\cos x + \sin x - 2}$$

Ex: $\boxed{\text{Find the form of } y_p}$

$$\underbrace{D^2(D+2)(D-1)(D^2+3)}_{L}y = 1 + e^x + \cos x$$

Solve $Ly=0$

$$y_c = c_1 + c_2 x + c_3 e^{-2x} + c_4 e^x + c_5 \cos\sqrt{3}x + c_6 \sin\sqrt{3}x$$

what annihilates $g(x)$?

$$\begin{array}{ccccc} 1 & + & e^x & + & \cos x \\ \uparrow & & \uparrow & & \uparrow \\ L_1 = (D) & & (D-1) & & (D^2+1) \end{array}$$

$$D(D-1)(D^2+1)\left[D^2(D+2)(D-1)(D^2+3)y\right] = D(D-1)(D^2+1)(1+e^x+\cos x) = 0$$

$$D^3(D+2)(D-1)^2(D^2+1)(D^2+3)y = 0$$

$$y = c_1 + c_2 x + c_3 x^2 + c_4 e^{-2x} + c_5 e^x + c_6 x e^x + c_7 \cos x + c_8 \sin x + c_9 \cos\sqrt{3}x + c_{10} \sin\sqrt{3}x$$

Thus,

$$y_p = Ax^2 + Bxe^x + C\cos x + E\sin x$$