$\boxed{4.3}$ Homogeneous Linear Equations with constant coefficients.

Find the general solution to

$$\sum_{k=0}^{n} a_k y^{(k)} = 0 \qquad (*)$$

All solutions to linear equations with constant coeff. are exponential functions or functions created from them.

Characteristic Equation

Try the solution $y = e^{mx}$ in $(*)$

$$y = e^{mx}$$
$$y' = me^{mx}$$
$$y'' = m^2 e^{mx}$$
$$y''' = m^3 e^{mx}$$
$$\vdots$$
$$y^{(k)} = m^k e^{mx}$$

so (*) becomes.

$$0 = \sum_{k=0}^{n} a_k y^{(k)} = \sum_{k=0}^{n} a_k m^k e^{mx}$$

$$= e^{mx} \sum_{k=0}^{n} a_k m^k = 0$$

Thus,

$$e^{mx} = 0 \quad \text{or} \quad \sum_{k=0}^{n} a_k m^k = 0$$

never happens.

$$(**) \quad a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0$$

(**) is called the characteristic equation. If we find the roots of the polynomial, then we will have solutions to the diff-eq. In general we have 4 cases.

$\boxed{\text{Consider } a_2 y'' + a_1 y' + a_0 y = 0}$

$\boxed{\text{Case I}}$ Distinct Real Roots

If both roots m_1 & m_2 are distinct real roots, then $e^{m_1 x}$ & $e^{m_2 x}$ are linearly independent. Hence, the solution is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$\boxed{\text{Case II}}$ - Repeated Real Roots.

Only one solution the polynomial exists.
thus, a second linearly indep. solution is

$$y_2 = y_1 \int \frac{e^{-\int p\,dx}}{y_1^2}\,dx = e^{m_1 x} \int \frac{e^{-\int \frac{a_1}{a_2}dx}}{e^{2m_1 x}} = e^{m_1 x} \int \frac{e^{-\frac{a_1}{a_2}x}}{e^{2m_1 x}}\,dx = e^{m_1 x} \int \frac{\cancel{e^{2m_1 x}}}{\cancel{e^{2m_1 x}}}\,dx$$

$$= x e^{m_1 x}$$

Note: $y'' + \frac{a_1}{a_2} y' + \frac{a_0}{a_2} = 0$

If there is only one solution,
then the discriminant must be zero.

$$m_1 = \frac{-a_1 \pm \sqrt{\underbrace{a_1^2 - 4a_0 a_2}_{\text{must be zero}}}}{2a_2} = \frac{-a_1}{2a_2} \Rightarrow 2m_1 = \frac{-a_1}{a_2}$$

So thus, the solution to
(***) $a_2 y'' + a_1 y' + a_0 y = 0$ is

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}$$

$\boxed{\text{Case III}}$ Complex Conjugate Roots.

If the roots to the charac. poly. are complex, then they are complex conjugates.

Hence $m_1 = \alpha + \beta i$

$m_2 = \alpha - \beta i$

So a general solution to (***) is

$$y = c_1 e^{(\alpha+\beta i)x} + c_2 e^{(\alpha-\beta i)x}$$

But this contains complex numbers. Let's change this to only depend on real numbers.

First, Euler's Formula is

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{(\alpha+\beta i)x} = e^{\alpha x} e^{\beta x i} = e^{\alpha x}(\cos\beta x + i\sin\beta x)$$

$$e^{(\alpha-\beta i)x} = e^{\alpha x} e^{-\beta x i} = e^{\alpha x}(\cos\beta x - i\sin\beta x)$$

Thus,

$$y = c_1 e^{(\alpha+\beta i)x} + c_2 e^{(\alpha-\beta i)x}$$

$$= c_1 e^{\alpha x}(\cos\beta x + i\sin\beta x) + c_2 e^{\alpha x}(\cos\beta x - i\sin\beta x)$$

$$= \underbrace{(c_1 + c_2)}_{c_1^*} e^{\alpha x}\cos\beta x + \underbrace{(c_1 i - c_2 i)}_{c_2^*} e^{\alpha x}\sin\beta x$$

Ex: $y'' + k^2 y = 0$ (2)

$m^2 + k^2 = 0 \Rightarrow m = \pm ki = \alpha + \beta i$

$\alpha = 0$

$\beta = k$

So the general solution is

$$y = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x$$

$$\boxed{y = c_1 \cos kx + c_2 \sin kx}$$

$y'' - k^2 y = 0$ (3)

$m^2 - k^2 = 0 \Rightarrow m = \pm k$

So the general solution is

$$y = c_1 e^{-kx} + c_2 e^{kx}$$

Let $c_1 = 1/2$, $c_2 = 1/2$.

$$y = \frac{e^{kx} + e^{-kx}}{2} = \cosh kx$$

Let $c_1 = -1/2$, $c_2 = 1/2$

$$y = \frac{e^{kx} - e^{-kx}}{2} = \sinh x.$$

An alternative solution is

$$y = c_1 \cosh kx + c_2 \sinh kx.$$