4.2 Constructing a second solution from a known solution

Reduction of order

Suppose $y_1(x)$ is a nonzero solution of the diff-eq

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \quad (*)$$

The process to find a second solution $y_2(x)$ consists of reducing the order of eq $(*)$ to a first order equation.

Ex: Note: $y_1 = e^x$ is a solution to $y'' - y = 0$.

Try to determine a solution of the form

$$y = u(x)e^x$$

$$y' = ue^x + u'e^x$$

$$y'' = ue^x + u'e^x + u'' + u'e^x$$

$$= ue^x + 2u'e^x + u''e^x$$

$$y'' - y = 0 = \cancel{ue^x} + 2u'e^x + u''e^x - \cancel{ue^x}$$

$$0 = (u'' + 2u')e^x$$

This means

$$u'' + 2u' = 0$$

Let $w = u'$, then

$$w' + 2w = 0$$

$$P(x) = 2 \Rightarrow \int P(x) = 2x \Rightarrow e^{\int P} = e^{2x}$$

$$e^{2x}w' + 2e^{2x}w = 0$$

$$\frac{d(e^{2x}w)}{dx} = 0$$

$$e^{2x}w = c$$

$$w = ce^{-2x}$$

$$u' = ce^{-2x}$$

$$u = -\frac{c_1}{2}e^{-2x} + c_2$$

So $y = ue^x = \left(-\frac{c_1}{2}e^{-2x} + c_2\right)e^x = -\frac{c_1}{2}e^{-x} + c_2 e^x$

A second solution is where $c_1 = -2, c_2 = 0 \Rightarrow y_2 = e^{-x}$

General case

$$y'' + P(x)y' + Q(x)y = 0. \qquad (1)$$

Suppose y_1 is a solution to (1)

$$y = u\,y_1$$

$$y' = u\,y_1' + u'y_1$$

$$y'' = u\,y_1'' + u'y_1' + u'y_1' + u''y_1$$

$$= u\,y_1'' + 2u'y_1' + u''y_1$$

Thus,

$$0 = y'' + P(x)y' + Q(x)y$$

$$= u\,y_1'' + 2u'y_1' + u''y_1$$

$$+ P(x)\left[u\,y_1' + u'y_1\right] + Q(x)\,u\,y_1$$

$$= u\underbrace{\left[y_1'' + P(x)y_1' + Q(x)y_1\right]}_{=0 \text{ (Since } y_1 \text{ is a solution to (1))}} + u'\left[2y_1' + P(x)y_1\right] + u''y_1$$

So

$$u'(2y' + Py_1) + u''y_1 = 0$$

$$w = u'$$

$$w'y_1 + w(2y_1' + Py_1) = 0$$

Note: this is seperable!

So $$-w'y_1 = w(2y_1' + Py_1)$$

$$-y_1 \frac{dw}{dx} = \left(2\frac{dy_1}{dx} + Py_1\right)w$$

$$\frac{dw}{w} = \left(-\frac{2}{y_1}\frac{dy_1}{dx} - P\right)dx$$

$$\int \frac{dw}{w} = \int -2\frac{dy_1}{y_1} - \int P\,dx$$

$$\ln|w| = -2\ln|y_1| - \int P\,dx + c_1$$

$$\ln|w| + \ln|y_1^2| = -\int P\,dx + c_1$$

$$\ln|wy_1^2| = -\int P\,dx + c_1$$

$$wy_1^2 = e^{-\int P\,dx + c_1} = ce^{-\int P(x)\,dx}$$

$$u' = w = \frac{1}{y_1^2}ce^{-\int P(x)\,dx}$$

$$u = \int \frac{c}{y_1^2} e^{-\int P(x)\,dx} + c_2$$

Thus, $$y = uy_1 = y_1 \int \frac{c_1 e^{-\int P\,dx}}{y_1^2}\,dx + c_2 y_1$$

Choose $c_1 = 1,\ c_2 = 0 \Rightarrow$

$$\boxed{y_2 = y_1 \int \frac{e^{-\int P\,dx}}{y_1^2}\,dx}$$

Ex: The function $y_1 = x^2$ is a solution of $x^2y'' - 3xy' + 4y = 0$.

Find the general solution on the interval $(0, \infty)$

$$y'' - \frac{3x}{x^2}y' + \frac{4}{x^2}y = 0$$

$$P(x) = -\frac{3}{x} \Rightarrow \int P(x)dx = -3\ln|x|$$

$$e^{-\int P(x)dx} = x^3$$

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2}dx$$

$$= x^2 \int \frac{x^3}{x^4}dx$$

$$= x^2 \left[\int \frac{dx}{x} + c \right]$$

$$y_2 = x^2 \ln x + cx^2$$

not needed

Note: No constants of integration are needed for this formula!

Thus, the general solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 x^2 + c_2 (x^2 \ln x + c x^2)$$

$$= c_1 x^2 + c_2 x^2 \ln x + c_2 c x^2$$

$$= (c_1 + c_2 c) x^2 + c_2 x^2 \ln x$$

$$= c_1^* x^2 + c_2 x^2 \ln x.$$

Note: It is easier if y_2 did not contain a constant in the integration part. Then

$$y = c_1 y_1 + c_2 y_2 = c_1 x^2 + c_2 x^2 \ln x$$

Ex: $x^2 y'' + x y' + y = 0 \Rightarrow x \in (0, \infty)$

$$\Rightarrow y'' + \frac{1}{x} y' + \frac{1}{x^2} y = 0 \Rightarrow P(x) = \frac{1}{x}$$

$y_1 = \cos(\ln x)$ is a solution to it.

$$-\int P(x)\,dx = -\int \frac{dx}{x} = -\ln|x| \Rightarrow e^{-\int P(x)dx} = \frac{1}{x}$$

$$y_2 = y_1 \int \frac{e^{-\int P dx}}{y_1^2}\,dx = \cos(\ln x) \int \frac{\frac{1}{x}\,dx}{\cos^2(\ln x)}$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$= \cos(\ln x) \int \frac{du}{\cos^2 u}$$

$$= \cos(\ln x)\tan(\ln x) = \sin(\ln x).$$

Thus, the general solution is $y = c_1 \cos(\ln x) + c_2 \sin(\ln x)$