3.3 Applications of Nonlinear Equations

Exponential Growth $\Rightarrow \frac{dP}{dt} = kP,\ k>0$

Logistic Equation

$$\frac{dP}{dt} = P(a+bP), \quad a, b, \text{ are pos const.}$$

Solution: logistic function

$$\frac{1}{P}\frac{dP}{dt} = \left(\begin{matrix}\text{average}\\ \text{birth rate}\end{matrix}\right) - \left(\begin{matrix}\text{average}\\ \text{death rate}\end{matrix}\right)$$

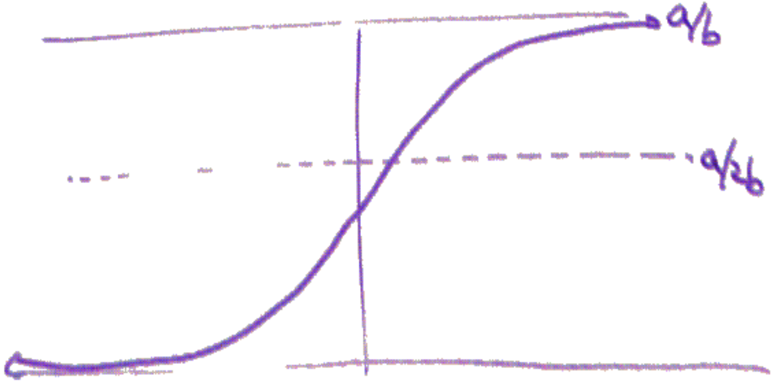
$$= a - bP$$

b is a pos const. of proport.

Quite accurate in predicting growth patterns in a limited space.

Solution: $$P(t) = \frac{ac_1 e^{at}}{1+bc_1 e^{at}} = \frac{ac_1}{bc_1 + e^{-at}}$$

with initial cond. $P(0) = P_0 \neq \frac{a}{b}$



Gompertz curves

A modification of logistic eq.

$$\frac{dP}{dt} = P(a - b\ln P)$$

$$\Rightarrow P(t) = e^{a/b} e^{-ce^{-bt}}$$

Chemical Reactions

$$\frac{dx}{dt} = kx \quad \text{(first order)}$$

$$\frac{dx}{dt} = k(\alpha - x)(\beta - x) \quad \text{(2nd order)}$$

Ex: A compound C is formed when two chemicals A & B are combined. The resulting reaction between the two chemicals is such that for each gram of A, 4 grams of B are used. It is observed that 30 g of compound C are formed in 10 minutes. Determine the amount of C at any time if the rate of the reaction is proportional to the amounts of A & B remaining and if there is initially 50 g of A and 32 g of B. How much of C is present at 15 min?

Interpret $t \to \infty$!

Sol: Let $x(t)$ denote the amount of C.

$X(0) = 0$

$X(10) = 30$

If x grams of C are formed from a grams of A
b grams of B.

$a+b = x \Rightarrow b = 4a \Rightarrow a+b = a+4a = x$

$5a = x$

$a = \frac{x}{5}$

$b = \frac{4x}{5}$

Start with 50 gram of A
32 grams of B

$$\frac{dx}{dt} = K_1\left(50 - \frac{x}{5}\right)\left(32 - \frac{4x}{5}\right)$$

$$= K\,(250-x)(40-x)$$

$$\int \frac{dx}{(250-x)(40-x)} = \int k\,dt$$

$$= kt + c$$

.... and then a miracle occurs....

$$\frac{250-x}{40-x} = ce^{210kt}$$

$$X(0) = 0 \Rightarrow \frac{250}{40} = c$$

$$x(10) = 30 \Rightarrow \frac{220}{10} = c^{210k(10)}$$

$$22 = ce^{2100k}$$

$$\frac{22}{c} = e^{2100k}$$

$$k = \frac{1}{2100} \ln\left(\frac{88}{25}\right) = .1258.$$

<u>Solve for x</u>:

...and then a miracle occurs....

$$x = \frac{250\left(e^{210kt} - 1\right)}{\frac{25}{4} e^{210kt} - 1}$$

$x(15) = ?$ → an intelligent reader can do this.

What is $\lim_{t \to \infty} x(t) = \frac{250}{\frac{25}{4}} = 40.$