$\boxed{3.2}$ Applications of Linear Equations.

Growth & Decay

$$\frac{dx}{dt} = kx, \quad x(t_0) = x_0$$

Ex: The pop of a town grows at a rate prop. to the popul. at time t. The initial pop of 500 increases by 15% in 10 years. What will the population be in 30 years.

$$\frac{dP}{dt} = kP \qquad P(0) = 500$$
$$P(10) = 1.15(500)$$

$$P = ce^{kt}$$

$$P(0) = 500 = ce^{k(0)} = c$$

$$P = 500e^{kt}$$

$$P(10) = 500e^{k(10)} = 500(1.15)$$

$$\boxed{k = \frac{1}{10}\ln 1.15}$$

$$P(30) = 500e^{\left(\frac{1}{10}\ln 1.15\right)30} = 761$$

Newton's Law of cooling

$$\frac{dT}{dt} = k(T - T_s)$$

$$T = T_s + ce^{kt}$$

Ex: A thermometer is removed from a room where the air temp is 70°F to the outside where the temp is 10°F. After $\frac{1}{2}$ min, the thermometer reads 50°C. What is the reading at t=1 minute?

$$T = T_s + ce^{kt}$$

$$T(0) = 70 = 10 + ce^0 \Rightarrow \boxed{c = 60}$$

$$T = 10 + 60e^{kt}$$

$$T(\tfrac{1}{2}) = 50 = 10 + 60e^{k\frac{1}{2}}$$

$$\frac{2}{3} = e^{\frac{k}{2}} \Rightarrow \boxed{k = 2\ln\frac{2}{3}}$$

$$T(1) = 10 + 60e^{(2\ln\frac{2}{3})1} =$$

$\boxed{3.2}$ Cont.

Mixture problem.

$$\frac{dA}{dt} = \begin{pmatrix}\text{rate of} \\ \text{substance entering}\end{pmatrix} - \begin{pmatrix}\text{rate of} \\ \text{substance leaving}\end{pmatrix}$$

$$= R_1 - R_2$$

Ex: A large tank is partially filled with 100 gallons of fluid in which 10 lbs of salt is dissolved. Brine containing $\frac{1}{2}$ lb of salt per gallon is pumped into the tank at a rate of 6 gal per min. The well-mixed solution is then pumped out at a slower rate of 4 gal per min. Find the number of lbs of salt in the tank after 30 min.

$$R_1 = \frac{6\text{ gal}}{\text{min}} \cdot \frac{\frac{1}{2}\text{ lb}}{\text{gal}} = \frac{3\text{ lb}}{\text{min}}$$

$$R_2 = \frac{4\text{ gal}}{\text{min}} \quad \frac{A}{100+2t}$$

$$\frac{dA}{dt} = 3 - \frac{4A}{100+2t}$$

$$\frac{dA}{dt} + \frac{2A}{50+t} = 3$$

$$P(t) = \frac{2}{50+t} \Rightarrow \int P(t)dt = 2\ln(50+t)$$

$$e^{\int P(t)dt} = e^{\ln(50+t)^2} = (50+t)^2$$

$$(50+t)^2\,dA + (50+t)^2 \frac{2}{50+t} A\,dt = 3(50+t)^2\,dt$$

$$d\left[(50+t)^2 A\right] = 3(50+t)^2\,dt$$

$$(50+t)^2 A = \int 3(50+t)^2\,dt = \frac{\cancel{3}(50+t)^3}{\cancel{3}} + c$$

$u = 50+t$
$du = dt$

$$A = 50+t + c(50+t)^{-2}$$

$$A(0) = 10 = 50 + \frac{c}{2500} \Rightarrow -40 = \frac{c}{2500}$$

$$c = 2500(-40) = -100\,000$$

$$A(t) = t+50 - \frac{100000}{(50+t)^2}$$

How much salt is in there in the long term?

$$\frac{A(t)}{100+2t} = \frac{t+50}{100+2t} \to \frac{1}{2} \quad - \quad \frac{100000}{2(50+t)^3} \to 0$$

$$A(30) = 30+50 - \frac{100000}{(50+30)^2}$$

$$= \frac{515}{8} = 64.375$$