2.5 Linear Equations

General:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

When $n=1$, we have a linear first-order diff-eq

$$(1) \qquad a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

Dividing both sides by $a_1(x)$ yields

$$(2) \qquad \boxed{\frac{dy}{dx} + P(x)y = f(x)}$$

Integrating factor

Using differentials, we can rewrite (2) as

$$dy + [P(x)y - f(x)]\,dx = 0$$

Linear equation posses the property than an integrating factor can be found

$$0 = \mu(x)dy + \mu(x)[P(x)y - f(x)]dx$$

So ↗ is an exact differential

$$\underbrace{\mu(x)}_{N(x,y)}\,dy + \underbrace{\mu(x)\left[P(x)y - f(x)\right]}_{M(x,y)}\,dx = 0$$

Exact if:

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

$$\frac{\partial \mu(x)}{\partial x} = \frac{\partial\left[\mu(x)\left[P(x)y - f(x)\right]\right]}{\partial y}$$

$$\frac{d\mu}{dx} = \frac{\partial\left[\mu(x)P(x)y\right]}{\partial y} - \cancel{\frac{\partial\left[\mu(x)f(x)\right]}{\partial y}}$$

$$\frac{d\mu}{dx} = \mu(x)P(x)$$

$$\int \frac{d\mu}{\mu} = \int P(x)\,dx$$

$$\ln|\mu| = \int P(x)\,dx$$

$$\boxed{\mu = e^{\int P(x)\,dx}}$$

Note: $f(x)$ wasn't involved in this.

So, from (1), we have

$$dy + [P(x)y - f(x)]\,dx = 0$$

$$dy + P(x)y\,dx = f(x)dx$$

$$\underbrace{e^{\int P(x)dx}dy + e^{\int P(x)dx}P(x)y\,dx} = e^{\int P(x)dx}f(x)\,dx$$

$$d(e^{\int P}y) = e^{\int P}f(x)dx$$

$$e^{\int P}y = \int e^{\int P}f(x)dx + c$$

$$y = e^{-\int P}\int e^{\int P}f(x)dx + ce^{-\int P}$$

Solve $\frac{dy}{dx} - 5y = 0$

$$P(x) = -5$$

$$\int P(x)dx = -5x \Rightarrow e^{-5x}$$

$$\underbrace{e^{-5x}dy - 5e^{-5x}y\,dx} = 0$$

$$d(e^{-5x}y) = 0$$

$$e^{-5x}y = c$$

$$\boxed{y = ce^{5x}}$$

Ex:

$$(x^2+x)\,dy = (x^5+3xy+3y)\,dx\;;\quad x>0$$

$$x(x+1)\,dy = \underbrace{x^5\,dx}_{x\text{'s}} + \underbrace{3xy\,dx + 3y\,dx}_{y\text{'s}}$$

$$x(x+1)\,dy - 3(x+1)y\,dx = x^5\,dx$$

$$dy - \frac{3}{x}y\,dx = \frac{x^5}{x(x+1)}\,dx$$

$$P(x) = -\frac{3}{x}$$

$$\int P(x) = -3\ln x = \ln x^{-3}$$

$$\mu = e^{\int P(x)dx} = e^{\ln x^{-3}} = x^{-3}$$

$$x^{-3}dy - \frac{3x^{-3}}{x}y\,dx = \frac{x^5x^{-3}}{x(x+1)}\,dx$$

$$\underbrace{x^{-3}dy - 3x^{-4}y\,dx}_{} = \frac{x}{x+1}\,dx$$

$$\int d(x^{-3}y) = \int \frac{x}{x+1}\,dx$$

$u = x+1$

$du = dx$

$$= \int \frac{u-1}{u}\,du$$

$$= \int 1 - \frac{1}{u}\,du$$

$$= u - \ln|u| + C_1$$

$$= x+1 - \ln|x+1| + C_1$$

$$x^{-3}y = x - \ln|x+1| + C$$

$$\boxed{y = x^4 - x^3\ln|x+1| + Cx^3}$$

$$\cos x\, y' + (\sin x) y = \cos^3 x, \quad y(0) = -1$$

$$y' + (\tan x) y = \cos^2 x$$

$$dy + (\tan x) y\, dx = \cos^2 x\, dx \checkmark$$

$$P(x) = \tan x$$

$$\int P(x)\, dx = \int \tan x\, dx = \int \frac{\sin x}{\cos x} dx = \int \frac{-du}{u} = -\ln|u| = -\ln|\cos x|$$

$u = \cos x$

$du = -\sin x\, dx$

Integrating factor is: $e^{-\ln|\cos x|} = e^{\ln|\cos x|^{-1}} = |\cos x|^{-1}$

initial value for x.

$\cos x > 0 \Rightarrow y(0) = -1$

$\hookrightarrow$ true where $-\frac{\pi}{2} < x < \frac{\pi}{2}$

integrating factor is $(\cos x)^{-1}$

$$(\cos x)^{-1} dy + (\tan x)(\cos x)^{-1} y\, dx = \cos^2 x (\cos x)^{-1} dx$$

$$(\cos x)^{-1} dy + \frac{\sin x}{\cos^2 x} y\, dx = \cos x\, dx$$

$$d\left((\cos x)^{-1} y\right) = \cos x\, dx$$

$$(\cos x)^{-1} y = \int \cos x\, dx$$

$$(\cos x)^{-1} y = \sin x + C$$

$$y = \sin x \cos x + C \cos x$$

$$y(0) = -1 = \sin 0 \cos 0 + c \cos 0$$

$$-1 = c$$

$$y = \sin x \cos x - \cos x$$