2.4 Exact Equations

Example:

$$x\,dy + y\,dx = 0$$

$$d(xy) = 0$$

$$\int d(xy) = \int 0$$

$$\boxed{xy = c}$$

Recall: the total differential of $z = f(x,y)$ is

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

If $f(x,y) = c$, then

$$0 = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

Suppose $\underbrace{x^2 - 5xy + y^3}_{f(x,y)} = c \quad (1)$

Then it follows that

$$(2x-5y)\,dx + (-5x+3y^2)\,dy = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x-5y}{5x-3y^2} \quad (2)$$

Can you go from (2) to (1)?

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DEF: A diff expression

$M(x,y)\,dx + N(x,y)\,dy$ is

an exact diff in a region R of the xy plane if it corresponds to the total derivative of some function f(x,y). A diff equation of the form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is said to be an exact equation if the expression on the left is an exact differential.

Ex: The equation

$x^2y^3\,dx + x^3y^2\,dy = 0$ is

exact since

$$d\left(\tfrac{1}{3}x^3y^3\right) = x^2y^3\,dx + x^3y^2\,dy$$

Criterion for an exact diff.

Thm: Let $M(x,y)$, $N(x,y)$ be cont & have cont. first partials in a reg. region R. Then a necessary & sufficient cond. that $M\,dx + N\,dy$ be an exact differential is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Note: this is saying (from 214)

$$\frac{\partial f}{\partial x \partial y} = \frac{\partial f}{\partial y \partial x}$$

Method of Solution

$M(x,y)\,dx + N(x,y)\,dy = 0$

1) Show $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

2) Assume $\dfrac{\partial f}{\partial x} = M(x,y)$

3) $f(x,y) = \left[\int M(x,y)\,dx\right] + \underbrace{g(y)}_{\text{const of integral}}$
(const. with respect to x).

4) Take derivative of 3) wrt y.

$$\frac{\partial f}{\partial y} = N(x,y) = \frac{\partial}{\partial y}\left[\int M(x,y)\,dx\right] + g'(y)$$

Hence, $g'(y) = N(x,y) - \dfrac{\partial}{\partial y}\int M\,dx \quad (*)$

5) Integrate $(*)$ wrt y & substitute into 3.

The solution is

$$\boxed{f(x,y) = c}$$

$\boxed{2.4}$ Cont.

EX: $(e^y + 2xy\cosh x)\frac{dy}{dx} + xy^2\sinh x + y^2\cosh x = 0$

$\underbrace{(xy^2\sinh x + y^2\cosh x)}_{M}\,dx + \underbrace{(e^y + 2xy\cosh x)}_{N}\,dy = 0$

$\frac{\partial M}{\partial y} = 2xy\sinh x + 2y\cosh x = \frac{\partial N}{\partial x} = 2y(x\sinh x + \cosh x)$

$= 2xy\sinh x + 2y\cosh x$

$f(x,y) = \left[\int M(x,y)\,dx\right] + g(y)$

$= \int xy^2\sinh x + y^2\cosh x\,dx + g(y)$

$u = x \quad dv = \sinh x$

$du = dx \quad v = \cosh x$

$= y^2\left[x\cosh x - \int \cosh x\,dx\right] + y^2\sinh x + g(y)$

$f(x,y) = xy^2\cosh x - y^2\sinh x + g(y)$

$+ y^2\sinh x$

$\frac{\partial f}{\partial y} = N(x,y) = 2xy\cosh x$

$-2y\sinh x + 2y\sinh x$

$+ g'(y)$

$= 2xy\cosh x + g'(y)$

$= e^y + 2xy\cosh x$

$\Rightarrow g'(y) = e^y$

$g(y) = e^y$

Thus $f(x,y) = xy^2\cosh x + e^y = c$

Thus, the solution is

$$f(x,y) = c$$

$$\boxed{xy^2\cosh x + e^y = c}$$

Ex: $x\frac{dy}{dx} = 2xe^x - y + 6x^2$

$$0 = \underbrace{(2xe^x - y + 6x^2)}dx - x\,dy$$

$$\frac{\partial M}{\partial y} = -1 = \frac{\partial N}{\partial x}$$

$$\frac{\partial f}{\partial x} = 2xe^x - y + 6x^2$$

$$f(x,y) = \int 2xe^x - y + 6x^2\,dx \quad + g(y)$$

$$= 2(xe^x - e^x) - xy + 2x^3 + g(y)$$

$$\frac{\partial f}{\partial y} = -x + g'(y) = N = -x$$

$$g'(y) = 0 \Rightarrow g(y) = c_1$$

$$f(x,y) = 2xe^x - 2e^x - xy + 2x^3 + c_1$$

Our solution is

$$f(x,y) = c \Rightarrow \boxed{2xe^x - 2e^x - xy + 2x^3 = c}$$

Integrating factor

It is sometimes possible to convert a non-exact differential equation into an exact one by multiplying by a function $\mu(x,y)$ called an integrating factor.

$$M(x,y)dx + N(x,y)dy = 0 \quad \text{not exact}$$

but $$\mu M(x,y)dx + \mu N(x,y)dy = 0 \quad \text{is exact.}$$

Ex: $$(x+y)\,dx + x\ln x\,dy = 0 \quad (*)$$

$(*)$ is not exact since $\dfrac{\partial M}{\partial y} = 1 \neq \dfrac{\partial N}{\partial x} = 1 + \ln x$

Let $\mu = \frac{1}{x}$, then

$$\left(1+\frac{y}{x}\right)dx + \ln x\,dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{1}{x} = \frac{\partial N}{\partial x}$$