2.3 Homogeneous Equations

DEF: If a function f has the property that

$$f(tx, ty) = t^n f(x, y)$$

for some real number n, then f is said to be a homogeneous function of degree n.

Ex: $f(x,y) = 3x^2y^2 + 2x^3y + 5xy^3 + 6x^4 + 7y^4$

$$f(tx,ty) = 3(tx)^2(ty)^2 + 2(tx)^3 ty + 5(tx)(ty)^3 + 6(tx)^4 + 7(ty)^4$$

$$= t^4(3x^2y^2 + 2x^3y + 5xy^3 + 6x^4 + 7y^4)$$

$$= t^4 f(x,y)$$

$f(x,y)$ is a homogeneous function of degree 4.

If $f(x,y)$ is homogeneous, then

$$f(x,y) = x^n f\left(1, \frac{y}{x}\right)$$

$$f(x,y) = y^n f\left(\frac{x}{y}, 1\right)$$

where $f\left(\frac{x}{y}, 1\right)$ and $f\left(1, \frac{y}{x}\right)$ are homogeneous functions of degree 0.

<u>Ex</u>: $f(x,y) = x^2 - 3xy + 5y^2 \Rightarrow$ degree 2.

$$= x^2 \underbrace{\left[1 - 3\frac{y}{x} + 5\left(\frac{y}{x}\right)^2\right]}_{f\left(1, \frac{y}{x}\right)}$$

<u>DEF</u>: <u>Homogeneous Equation</u>

A diff-eq of the form

(*) $M(x,y)\,dx + N(x,y)\,dy = 0$ is said to be homogeneous if M & N are homogeneous functions.

Note: to solve $(*)$, we make the substitution

$$y = ux \quad (\text{or } x = vy)$$

$$dy = u\,dx + x\,du$$

$$\begin{aligned}
0 &= M(x,y)\,dx + N(x,y)\,dy = \\
&= M(x,ux)\,dx + N(x,ux)(u\,dx + x\,du) \\
&= [\underbrace{M(x,ux)} + u\underbrace{N(x,ux)}]\,dx + xN(x,ux)\,du \\
&= [x^n M(1,u) + ux^n N(1,u)]\,dx + x\,x^n N(1,u)\,du
\end{aligned}$$

$$\to [M(1,u) + uN(1,u)]\,dx = -xN(1,u)\,du$$

$$\boxed{-\frac{dx}{x} = \frac{N(1,u)}{M(1,u) + uN(1,u)}\,du}$$

Solve: $\dfrac{dy}{dx} = \dfrac{y-x}{y+x}$

$$(y+x)\,dy = (y-x)\,dx$$

$$0 = (y-x)\,dx - (y+x)\,dy$$

$$\left.\begin{aligned} M(x,y) &= y-x \\ N(x,y) &= y+x \end{aligned}\right\}\ \text{degree 1 homog.}$$

$y = ux$

$dy = u\,dx + x\,du$

$$0 = (ux - x)\,dx - (ux + x)(u\,dx + x\,du)$$

$$= (u-1)x\,dx - u(u+1)x\,dx - (u+1)x^2\,du$$

$$= [u - 1 - u(u+1)]\,dx - (u+1)x\,du$$

$$= [-u^2 - 1]\,dx - (u+1)x\,du$$

$$(u+1)x\,du = [-u^2 - 1]\,dx$$

$$\frac{dx}{x} = -\frac{u+1}{u^2+1} \qquad dy$$

$$\int \frac{dx}{x} = -\int \frac{u+1}{u^2+1}\,du$$

$$\ln|x| = -\frac{1}{2}\int \frac{2u}{u^2+1}\,du - \int \frac{1}{u^2+1}\,du$$

$w = u^2 + 1$

$dw = 2u\,du$

$$\boxed{\ln|x| = -\frac{1}{2}\ln(u^2+1) - \tan^{-1} u + c}$$

$$= -\frac{1}{2}\ln\left(\frac{y^2}{x^2} + 1\right) - \tan^{-1}\left(\frac{y}{x}\right) + c$$

$$= -\frac{1}{2}\ln(x^2 + y^2) + \frac{1}{2}\ln(x^2) - \tan^{-1}\left(\frac{y}{x}\right) + c$$

$$+ \ln|x|$$

Note: A homogeneous equation can also be written as

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

Proof:

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$\frac{dy}{dx} = \frac{-M(x,y)}{N(x,y)} = \frac{-\cancel{x^n} M\left(1,\frac{y}{x}\right)}{\cancel{x^n} N\left(1,\frac{y}{x}\right)}$$

$$= -\frac{M\left(1,\frac{y}{x}\right)}{N\left(1,\frac{y}{x}\right)} = F\left(\frac{y}{x}\right)$$

$$\frac{du}{dx} = F\left(\frac{y}{x}\right)$$

$$y = ux$$

$$\frac{dy}{dx} = u + x\frac{du}{dx}$$

$$u + x\frac{du}{dx} = F\left(\frac{ux}{x}\right) = F(u)$$

$$x\frac{du}{dx} = F(u) - u$$

$$\boxed{\frac{du}{F(u)-u} = \frac{dx}{x}}$$

$$2x^2 \frac{dy}{dx} = 3xy + y^2 \qquad y(1)=2$$

$$2x^2\,dy = (3xy + y^2)\,dx$$

$$0 = (3xy + y^2)\,dx - 2x^2\,dy$$

$$y = ux$$

$$dy = u\,dx + x\,du$$

$$0 = \left(3x(ux) + (ux)^2\right)dx - 2x^2(u\,dx + x\,du)$$

$$= (3ux^2 + u^2x^2)\,dx - 2x^2u\,dx - 2x^3\,du$$

$$= (3u + u^2)\,dx - 2u\,dx - 2x\,du$$

$$2x\,du = (3\underbrace{u + u^2 - 2u})\,dx$$

$$\int \frac{2\,du}{u^2 + u} = \int \frac{dx}{x}$$

$$\Rightarrow \ln|x| = \int \frac{2}{u(u+1)}$$

$$= 2\int \frac{1}{u} - \frac{1}{u+1}\,du$$

$$\Rightarrow \ln|x| = 2\ln|u| - 2\ln|u+1| + C$$

$$\ln|x| = 2\ln\left|\frac{u}{u+1}\right| + C$$

$$\ln|x| = 2\ln\left|\frac{y/x}{y/x+1}\right| + c$$

$$\ln|x| = 2\ln\left|\frac{y}{y+x}\right| + c$$

$$\ln|1| = 2\ln\left|\frac{-2}{-2+1}\right| + c$$

$$0 = 2\ln|2| + c$$

$$c = -2\ln 2$$

$$\boxed{\ln|x| = 2\ln\left|\frac{y}{y+x}\right| - 2\ln 2}$$

$$\ln|x| = 2\ln\left|\frac{y}{y+x}\frac{1}{2}\right|$$

$$x = \left(\frac{1}{2}\frac{y}{y+x}\right)^2 = \frac{y^2}{4(y+x)^2}$$

$$y^2 = 4x(y+x)^2$$