$\boxed{2.2}$ Separable Variables.

Solve IVP's of the form

$$(*) \qquad \frac{dy}{dx} = \frac{g(x)}{h(y)} \quad \text{subject to } y(x_0) = y_0$$

So the solution is

$$h(y)\,dy = g(x)\,dx$$

$$\int h(y)\,dy = \int g(x)\,dx$$

$$H(y) = G(x) + C$$

If H^{-1} exists, then

$$y = H^{-1}(G(x) + C)$$

$$y(x_0) = y_0$$

$$H(y_0) = G(x_0) + C$$

$$C = H(y_0) - G(x_0)$$

Ex: $dx - x^2 dy = 0$ subject to $y(1) = 0$

$$dx = x^2 dy$$

$$\int x^{-2} dx = \int dy$$

$$\frac{x^{-1}}{-1} = y + c$$

$$y = -\frac{1}{x} + c_1$$

$$y(1) = 0 = -1 + c_1$$

$$\boxed{1 = c_1}$$

$$\boxed{y = -\frac{1}{x} + 1} = -x^{-1}$$

Show it satisfies $\frac{dy}{dx} = \frac{1}{x^2}$

$$\frac{dy}{dx} = 1x^{-2} = \frac{1}{x^2} \checkmark$$

Same thing!

Ex:

$$(e^{-y}+1)\sin x\,dx-(1+\cos x)\,dy=0$$

$$(e^{-y}+1)\sin x\,dx = (1+\cos x)\,dy$$

$$\frac{\sin x}{1+\cos x}\,dx = \frac{1}{e^{-y}+1}\,dy$$

$$\int \qquad \qquad \int$$

$y(0)=0$

Ex: $(e^{-y}+1)\sin x\,dx - (1+\cos x)\,dy = 0$

$(e^{-y}+1)\sin x\,dx = (1+\cos x)\,dy$

$$\frac{\sin x}{1+\cos x}\,dx = \frac{1}{e^{-y}+1}\,dy$$

$$\int \frac{\sin x}{1+\cos x}\,dx = \int \frac{e^{y}}{1+e^{y}}\,dy$$

$u = 1+\cos x$

$du = -\sin x\,dx$

$w = 1+e^{y}$

$dw = e^{y}dy$

$$\int \frac{-du}{u} = \int \frac{dw}{w}$$

$\Rightarrow -\ln|u| = \ln|w| + C$

$-\ln|1+\cos x| = \ln|1+e^{y}| + C$

$-\ln|1+1| = \ln|1+1| + C$

$\boxed{-2\ln 2 = C}$

Our solution is

$$-\ln|1+\cos x| = \ln|1+e^y| - 2\ln 2$$

$$-\ln|1+\cos x| + 2\ln 2 = \ln(1+e^y)$$

$$-\ln|1+\cos x| + \ln 4$$

$$\ln\left|\frac{4}{1+\cos x}\right| = \ln(1+e^y)$$

$$\left|\frac{4}{1+\cos x}\right| = 1+e^y$$

$$\boxed{y = \ln\left(\frac{4}{1+\cos x} - 1\right)}$$

$$-1 \le \cos x \le 1$$

$$0 < 1+\cos x < 2$$