Differential Equations

1.1 Basic Terminology.

DEF: differential equations.

An equation containing the derivatives or differentials of one or more dependent vars

Ex:

$$\frac{dy}{dx} = x^2 + y^2$$

Classified by type, order, and linearity.

Classification by type

If the eq. only contains ordinary derivatives with respect to a single dependent variable, it is said to be an ordinary differential equation (ODE).

$$(y-x)dx + 4x\,dy = 0$$

$$\frac{dy}{dx} - \frac{dv}{dx} = 0 + x$$

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 6y = 0$$

(PDE) Partial differential equations

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{or} \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = u$$

Classification by Order

The order of the highest-order derivative in a differential equation is called the order of the equation

$$\underbrace{\frac{d^2y}{dx^2}}_{\text{second order}} + \underbrace{5\left(\underbrace{\frac{dy}{dx}}_{\text{first order}}\right)^3 - 4y = e^x}_{\text{second order}}$$

We'll study ODE's first

A general n^{th} order ODE is often represented as

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}\right) = 0$$

The following is a special case:

A differential equation is said to be linear if it can be written in the form

$$(*) \quad a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$$\begin{bmatrix} a_n(x) & a_{n-1}(x) & \cdots & a_0(x) \end{bmatrix} \begin{bmatrix} \frac{d^ny}{dx^n} \\ \vdots \\ y \end{bmatrix} = g(x)$$

In terms of linear algebra,

$$a(x)^T Y(y) = g(x)$$

$$a(x) = \begin{bmatrix} a_n(x) \\ a_{n-1}(x) \\ \vdots \\ \vdots \\ a_0(x) \end{bmatrix} \qquad Y(y) = \begin{bmatrix} \frac{d^n y}{dx^n} \\ \frac{d^{n-1} y}{dx^{n}} \\ \vdots \\ \frac{dy}{dx} \\ y \end{bmatrix}$$

Nonlinear diff-eq

$$\underline{y}\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right) = x \qquad \frac{d^3y}{dx^3} + \underline{y^2} = 0$$

nonlinear nonlinear

DEF: Any function f defined on some interval I, which, when substituted into a diff-eq reduces the eq. to an identity, is said to be a solution to the diff-eq.

Ex: Verify $y = \frac{x^4}{16}$ (**) is a sol to

(***) $\frac{dy}{dx} = xy^{1/2}$ on the interval $(-\infty, \infty)$

$$y = \frac{x^4}{16}$$

deriv

$$(**) \quad \frac{dy}{dx} = \frac{4x^3}{16} = \frac{x^3}{4}$$

$$(***) \quad \frac{dy}{dx} = x y^{1/2} = x\left(\frac{x^4}{16}\right)^{1/2}$$

$$= \frac{x^3}{4}$$

$$(**) \quad = \frac{x^3}{4}$$

Ex:

$y = xe^x$ satisfies $y'' - 2y' + y = 0$

on the interval $(-\infty, \infty)$

$$y' = xe^x + e^x$$

$$y'' = xe^x + e^x + e^x = xe^x + 2e^x$$

$$y'' - 2y' + y = 0$$

$$\cancel{xe^x} + \cancel{2e^x} - \cancel{2xe^x} - \cancel{2e^x} + \cancel{xe^x} = 0$$

Note: for both examples, $y=0$ is also a solution!

$y=0$ is called the "trivial" solution.

Explicit vs. Implicit Solutions.

If you can write the solution of an ODE in the form

$$y = f(x), \text{ then}$$

y is said to be an explicit solution.

Ex: An implicit solution

For $-2 < x < 2$ $\quad x^2 + y^2 - 4 = 0$

Is an implicit solution to the diff-eq

$$\frac{dy}{dx} = -\frac{x}{y}$$

By implicit differentiation,

$$\frac{d}{dx}\left(x^2 + y^2 - 4\right) = \frac{d}{dx}(0)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x}{y}$$

Note:

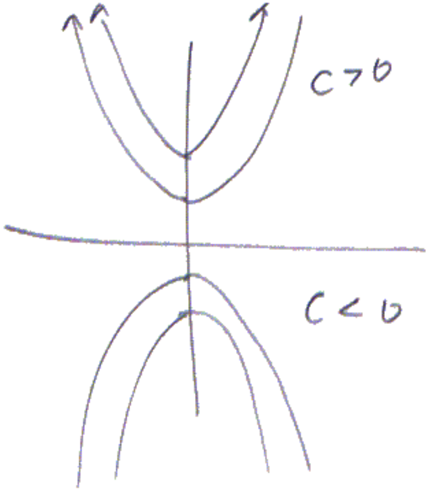
$$y = \sqrt{4 - x^2}$$

$$y = -\sqrt{4 - x^2}$$

are explicit solutions

Note that a diffeq usually possesses an infinite number of solutions

$y = ce^{x^2}$ where c is arbitrary



Ex: $y = c_1 \cos 4x$

$y = c_2 \sin 4x$ $\Rightarrow$ soln to $y'' + 16y = 0$

$\boxed{y = c_1 \cos 4x + c_2 \sin 4x \text{ is also a soln}}$

$y'' = -16c_1 \cos 4x$

$y'' + 16y = \cancel{-16c_1 \cos 4x + 16c_1 \cos 4x} = 0$

Ex: Verify

$y_1 = c_1 \cos 4x$ and

$y_2 = c_2 \sin 4x$ are

solutions to the diff-eq

$$y'' + 16y = 0$$

$$y_1' = -4c_1 \sin 4x$$
$$y_1'' = -16c_1 \cos 4x$$

$$y_1'' + 16y_1 = -16c_1\cos 4x + 16c_1 \cos 4x$$
$$= 0$$

$$y_2' = 4c_2 \cos 4x$$
$$y_2'' = -16c_2 \sin 4x$$

$$y_2'' + 16y_2 = -16c_2 \sin 4x + 16c_2 \sin 4x = 0$$

$$y = y_1 + y_2$$
$$= c_1 \cos 4x + c_2 \sin 4x$$

$$y'' + 16y = y_1'' + y_2'' + 16y_1 + 16y_2$$
$$= \underbrace{(y_1'' + 16y_1)}_{0} + \underbrace{(y_2'' + 16y_2)}_{0}$$
$$= 0$$

Ex: A Piece-wise defined solution

Any function in the one-parameter family $\boxed{y = cx^4}$ is a solution of the diff-eq $\boxed{xy' - 4y = 0}$

The piecewise-defined

$$y = \begin{cases} -x^4, & x < 0 \\ x^4, & x \geq 0 \end{cases}$$

is also a solution.

Note: the study of diff-eq is similar to integral calculus.

In integral calculus, we get a family of curves when finding an indefinite integral.

Similarly, when solving the diff-eq

$F(x, y, y') = 0$, we get a family of curves $G(x, y, c) = 0$ containing one arbitrary constant.

Generally, when solving $F(x, y, y', y'', \cdots, y^{(n)}) = 0$ we get an n-parameter family

$$G(x, y, c_1, c_2, \cdots, c_n) = 0$$

A solution that is free of arbitrary constants is called a particular solution.

Sometimes a solution to an ODE cannot be found by varying the arbitrary constant. Such a solution is called a singular solution.

Ex: The general sol'n to $y' = xy^{1/2}$ is given by

$$y = \left(\frac{1}{4}x^2 + c\right)^2$$

Note that $y=0$ is also a solution, but you can't get it by varying c.

$\boxed{1.2}$ Some Mathematical Models.

Free Falling bodies

$$\frac{d^2s}{dt^2} = -g$$

$s(0) = s_0$ (initial height)

$s'(0) = v_0$ (initial velocity)