

$$\text{Median } E(M) = \mu$$

$$\text{Midrange } E\left(\frac{X_{1:n} + X_{n:1}}{2}\right) = \mu$$

$$E(X_i) = \mu$$

Note: Sometimes an estimator is only off by a const.

$$\hat{\sigma}_{mle}^2 = \frac{1}{n} \sum (X_i - \bar{x})^2 = \frac{n-1}{n} s^2$$

$$E(\hat{\sigma}_{mle}^2) = \frac{n-1}{n} E(s^2) = \frac{n-1}{n} \sigma^2$$

$$\hat{\theta}_{mle} = \frac{n}{n-1} \hat{\sigma}_{mle}^2$$

$$E(\hat{\theta}_{mle}) = \frac{n}{n-1} E(\hat{\sigma}_{mle}^2) = \frac{n}{n-1} \frac{n-1}{n} \sigma^2 = \sigma^2$$

Uniformly Minimum Variance Unbiased Estimators (UMVUE)

DEF: Let X_1, X_2, \dots, X_n be a RS from $f(x; \theta)$

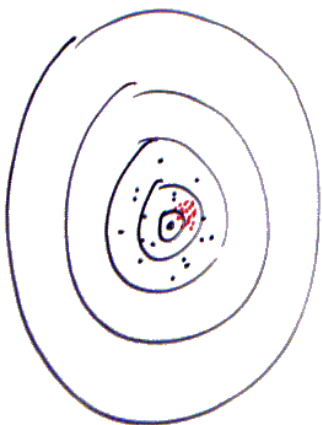
An estimator T^* of $\tau(\theta)$ is called the UMVUE of $\tau(\theta)$ if.

1. T^* is unbiased for $\tau(\theta)$
2. For any other unbiased est T of $\tau(\theta)$

$$\text{Var}(T^*) \leq \text{Var}(T) \text{ for all } \theta \in \Omega$$

If T is an unbiased est of $\tau(\theta)$, then the Cramer-Rao lower bound (CRLB) is

$$\text{Var}(T) \geq \frac{[\tau'(\theta)]^2}{E \left\{ \left[\frac{\partial}{\partial \theta} \ln f(x; \theta) \right]^2 \right\}}$$



DEF: If T is an est of $\tau(\theta)$,
then the bias is given by

$$b(T) = E(T) - \tau(\theta) \\ = E(T - \tau(\theta))$$

An unbiased est, T , for $\tau(\theta)$
has the prop:

$$E(T) = \tau(\theta)$$

$$b(T) = E(T) - \tau(\theta) = 0.$$

the mean-squared error (MSE)
is given by

$$MSE(T) = E[T - \tau(\theta)]^2$$

Thm: If T is an est of $\tau(\theta)$,

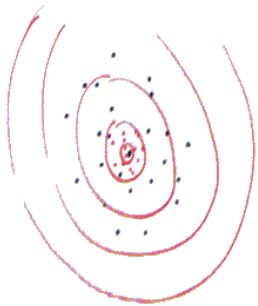
then

$$MSE(T) = \text{var}(T) + [b(T)]^2$$

$$\sigma_{MLE}^2 = \frac{n-1}{n} \sigma^2$$

$$\lim_{n \rightarrow \infty} \sigma_{MLE}^2 = \sigma^2$$

One idea for choosing an estimator is to choose the one that tends to be closest or "most concentrated" around the true value.



It might be reasonable to say that T_1 is better than T_2 if

$$P[\tau(\theta) - \varepsilon < T_1 < \tau(\theta) + \varepsilon] \geq P[\tau(\theta) - \varepsilon < T_2 < \tau(\theta) + \varepsilon]$$

for all $\varepsilon > 0$,

Note: By Chebyshev's

$$P[\tau(\theta) - \varepsilon < T < \tau(\theta) + \varepsilon] = P[-\varepsilon < T - \tau(\theta) < \varepsilon] = P[|T - \tau(\theta)| < \varepsilon]$$
$$\geq 1 - \frac{\text{Var}(T - \tau(\theta))}{\varepsilon^2} = 1 - \frac{\text{Var}(T)}{\varepsilon^2}$$

Our goal: Pick T^* such that $\text{Var}(T^*) \leq \text{Var}(T)$

Note: If proper differentiability cond. holds, then it can be shown

$$E\left[\frac{\partial}{\partial \theta} \ln f(x; \theta)\right]^2 = -E\left[\frac{\partial^2}{\partial \theta^2} \ln f(x; \theta)\right]$$

Ex: PS from $X_i \sim \text{EXP}(\theta)$.

$$\ln f(x; \theta) = -\frac{x}{\theta} - \ln \theta$$

$$\frac{\partial \ln f(x; \theta)}{\partial \theta} = \frac{x}{\theta^2} - \frac{1}{\theta} = \frac{x - \theta}{\theta^2}$$

$$\begin{aligned} E\left[\frac{\partial}{\partial \theta} \ln f(x; \theta)\right] &= E\left[\frac{(x - \theta)^2}{\theta^4}\right] = \frac{1}{\theta^4} E(x - \theta)^2 \\ &= \frac{\text{Var}(x)}{\theta^4} = \frac{\theta^2}{\theta^4} = \frac{1}{\theta^2} \end{aligned}$$

CRLB is: $E(x) = \theta \Rightarrow E(\bar{x}) = \mu = \theta$ $T^* = \bar{x}$
 $\pi(\theta) = \theta$

$$\text{Var}(T) = \frac{[\gamma(\theta)]^2}{n E\left[\frac{\partial}{\partial \theta} \ln f(x; \theta)\right]^2} = \frac{1}{n \cdot \frac{1}{\theta^2}} = \frac{\theta^2}{n}$$

$$\begin{aligned} E(\bar{x}) &= \mu \\ \text{Var}(\bar{x}) &= \frac{\sigma^2}{n} = \frac{\theta^2}{n} \end{aligned}$$

\bar{x} is the UMVUE of θ .

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 $\eta(\theta) = \theta$

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