8.4 The t, F, and Beta Dist.

Certain functions of normal samples are very important.

$$\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

Student's t dist.

Motivation

Note: s^2 can be used to make inferences about σ^2

puts

$\bar{X}$ can be used to make inferences about μ.

Thm: If $Z \sim N(0,1)$

$C \sim \chi^2(\nu)$

$Z \perp\!\!\!\perp C$, then

$$T = \frac{Z}{\sqrt{\frac{C}{\nu}}}$$

is referred to as Student's t dist with degrees of freedom ν.

$$T \sim t(\nu)$$

Proof: The joint density of $\underset{\sim N(0,1)}{Z}, \underset{\sim \chi^2(\nu)}{C}$ is

$$f_{Z,C}(z,c) = \frac{1}{2^{\nu/2}\Gamma(\frac{\nu}{2})\sqrt{2\pi}} c^{\frac{\nu}{2}-1} e^{-\frac{1}{2}(z^2+c)}$$

Make the transform $T = \dfrac{Z}{\sqrt{\frac{C}{\nu}}}$, $W = C$

$$Z = T\sqrt{\frac{W}{\nu}}, \quad C = W$$

$$|J| = \begin{vmatrix} \frac{\partial c}{\partial w} & \frac{\partial c}{\partial T} \\ \frac{\partial z}{\partial w} & \frac{\partial z}{\partial T} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \sim & \sqrt{\frac{W}{\nu}} \end{vmatrix} = \sqrt{\frac{W}{\nu}}$$

$$f_{T,W}(t,w) = f_{Z,C}(z,c)\,|J|$$

$$= f_{Z,C}\left(t\sqrt{\frac{w}{\nu}}, w\right)\sqrt{\frac{w}{\nu}}$$

$$f_{T,W}(t,w) = c \cdot w^{\frac{\nu}{2}-1} e^{-\frac{1}{2}\left(\frac{t^2 w}{\nu}+w\right)} \sqrt{\frac{w}{\nu}}$$

$$f_T(t) = \int_0^\infty f_{T,W}(t,w)\,dw$$

$$= \frac{c}{\sqrt{\nu}} \int_0^\infty w^{\left(\frac{\nu-1}{2}\right)} e^{-\frac{1}{2}w\left(\frac{t^2}{\nu}+1\right)}\,dw$$

$$u = \frac{1}{2}\left(\frac{t^2}{\nu}+1\right)w$$

$$du = \frac{1}{2}\left(\frac{t^2}{\nu}+1\right)dw$$

Snedecor's F dist

Thm: If $V_1 \sim \chi^2(\nu_1)$ and $V_2 \sim \chi^2(\nu_2)$, and $V_1 \perp\!\!\!\perp V_2$, then

$$F = \frac{(V_1/\nu_1)}{(V_2/\nu_2)} = \frac{V_1}{\nu_1} \cdot \frac{\nu_2}{V_2} = \frac{\nu_2}{\nu_1} V_1 V_2^{-1}$$

is distributed as Snedecor's F dist with ν_1 and ν_2 degrees of freedom

ν_1 = numerator degrees of freedom

ν_2 = denominator degrees of freedom.

$$f(x) = \frac{\Gamma\left(\frac{\nu_1+\nu_2}{2}\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)} \left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} x^{\frac{\nu_1}{2}-1}\left(1+\frac{\nu_1}{\nu_2}x\right)^{-\left(\frac{\nu_1+\nu_2}{2}\right)}$$

Thm: If $X \sim F(\nu_1, \nu_2)$,

then

$$E[X^r] = \frac{\left(\frac{\nu_2}{\nu_1}\right)^r \Gamma\left(\frac{\nu_1}{2}+r\right)\Gamma\left(\frac{\nu_2}{2}-r\right)}{\Gamma\left(\frac{\nu_1}{2}\right)\Gamma\left(\frac{\nu_2}{2}\right)}$$

$\nu_2 > 2r$

$$E[X] = \frac{\nu_2}{\nu_2-2}, \quad \nu_2 > 2.$$

$$Var(X) = \frac{2\nu_2^2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-2)^2(\nu_2-4)}, \quad \nu_2 > 4$$

Ex: Let $X_1, X_2, \dots, X_{n_1}$
$Y_1, Y_2, \dots, Y_{n_2}$ indep R.S.
from a pop with $X_i \sim N(\mu_1, \sigma_1^2)$
$Y_j \sim N(\mu_2, \sigma_2^2)$

$\nu_1 = n_1 - 1$, $\nu_2 = n_2 - 1$

Then
$$\frac{\nu_1 S_1^2}{\sigma_1^2} \sim \chi^2(\nu_1) \qquad \frac{\nu_2 S_2^2}{\sigma_2^2} \sim \chi^2(\nu_2)$$

$$\frac{\left(\frac{\cancel{\nu_1} S_1^2}{\sigma_1^2}\right)/\cancel{\nu_1}}{\left(\frac{\cancel{\nu_2} S_2^2}{\sigma_2^2}\right)/\cancel{\nu_2}} = \frac{\sigma_2^2 S_1^2}{\sigma_1^2 S_2^2} \sim F(\nu_1, \nu_2)$$

Beta Distribution

If $X \sim F(\nu_1, \nu_2)$, then

$$Y = \frac{\frac{\nu_1}{\nu_2} X}{1 + \frac{\nu_1}{\nu_2} X} \sim BETA\left(\frac{\nu_1}{2}, \frac{\nu_2}{2}\right)$$

If $W \sim BETA(\alpha, \beta)$

$$f_W(w) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} w^{\alpha-1}(1-w)^{\beta-1}$$
$$\times I_{(0,1)}(w)$$