$\boxed{8.3}$ Sampling Dist.

A statistic is a random var.
It has a dist which depends on the R.S.
this dist is called the sampling dist.

Linear Combinations of Normal R.V.

Thm: If $X_i \sim N(\mu_i, \sigma_i^2)$ denote independent R.V. It follows that

$$Y = \sum_{i=1}^{n} a_i X_i \sim N\left(\sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} a_i^2 \sigma_i^2\right)$$

Proof: Use MGF's.

Corollary:

If $X_1, \dots, X_n$ is RS $N(\mu, \sigma^2)$

then
$$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

Note: compare to CLT

$$\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} \xrightarrow{d} N(0,1)$$

$$\bar{X} \mathrel{\dot\sim} N\left(\mu, \frac{\sigma^2}{n}\right)$$

Chi-Squared dist   "chi" is pronounced like "Kite"

χ^2

nu ν, not V, but ν

If $Y \sim GAM(2, \frac{\nu}{2})$, the Y is said to have a Chi-Squared dist.

ν = degrees of freedom

$Y \sim \chi^2(\nu)$

Thm: If $Y \sim \chi^2(\nu)$

$M_Y(t) = (1-2t)^{-\nu/2}$

$E[Y^r] = \frac{2^r \Gamma(\frac{\nu}{2}+r)}{\Gamma(\frac{\nu}{2})}, r > -\frac{\nu}{2}$

$E(Y) = \nu$

$V(Y) = 2\nu$

Thm: If $X \sim GAM(\theta, \kappa)$, then

$Y = \frac{2X}{\theta} \sim \chi^2(2\kappa)$

We can use a χ^2-table to compu the CDF for any Gamma dist.

$F_X(x) = F_Y(\frac{2x}{\theta}, 2\kappa)$

Ex: X = time in years until failure of a comp.

$X \sim GAM(\overset{\theta,\,K}{3,2})$

Find the 10^{th} Percentile.

$$P[X \le \underbrace{X_{0.10}}_{10^{th}\text{ percentile}}] = F_Y\left(\boxed{\frac{2X_{0.10}}{\theta}}\right) = .10$$

$$\frac{2X_{0.10}}{\theta} = \chi^2_{0.10}(2K)$$

$$\frac{2}{3}X_{0.10} = \chi^2_{0.10}(4) = 1.06$$

$$X_{0.10} = \frac{3}{2}(1.06) = 1.59$$

Thm: If $Y_i \sim \chi^2(\nu_i)$ and Y_i are ind

then $V = \sum_{i=1}^{n} Y_i \sim \chi^2\left(\sum_{i=1}^{n} \nu_i\right)$

Proof: Use MGF.

Thm: If $Z \sim N(0,1)$,

then $Z^2 \sim \chi^2(1)$

Proof: Transform to $Y = Z^2$

Corollary If $X_1, \ldots, X_n$ is a RS from $N(\mu, \sigma^2)$

$$\sum_{i=1}^{n} \frac{(X_i-\mu)^2}{\sigma^2} \sim \chi^2(n)$$

$$\frac{n(\bar{X}-\mu)^2}{\sigma^2} \sim \chi^2(1)$$

Proof:

$$\bar{X} \sim N(\mu, \sigma^2/n) \Rightarrow \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1) \Rightarrow \left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2 \sim \chi^2(1)$$

$$\left(\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\right)^2 = \frac{n(\bar{X}-\mu)^2}{\sigma^2}$$

linear comb of normal R.V.'s.

$$Y = \sum a_i X_i \sim N\left(\sum a_i \mu_i, \sum a_i^2 \sigma_i^2\right)$$

$$\bar{X} = \frac{1}{n}\sum X_i \Rightarrow a_i = \frac{1}{n}$$

What is the dist of

$$S^2 = \sum_{i=1}^{n} \frac{(X_i-\bar{X})^2}{n-1}$$

Difficult: $X_i - \bar{X}$ is not indep of $X_j - \bar{X}$.

$$\sum (X_i - \bar{X}) = 0.$$

Thm: If $X_1, \ldots, X_n$ denote a RS. from $N(\mu, \sigma^2)$

1. $\bar{X}$ and $X_i - \bar{X}$ are independent
2. $\bar{X}$ and S^2 are independent
3. $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1)$